Cambridge Assessment
Admissions Testing

Test of Mathematics for University Admission

Notes on Logic and Proof

November 2021

Notes for Paper 2 of
The Test of Mathematics for University Admission

You should check the website regularly for updates to these notes.

Introduction

The formal side of mathematics – that of theorems and proofs – is a major part of the subject and is the main focus of Paper 2. These notes are intended to be a brief introduction to the ideas involved, for the benefit of candidates who have not yet met them within their mathematics classes or within their wider mathematical reading.

Mathematics, in part, is about working out the relationships between (mathematical) statements. And it is very important that everyone (that is, all mathematicians) write and talk about these relationships in the same standard way. It is important because mathematics is expressed formally using a rigorous language and if different people mean different things when they use the same words, then it would be difficult to ensure that everyone is talking about (and agreeing about) the very same things. Learning the basics of the rules and terms used by mathematicians is essential if you are to be able to understand and contribute to mathematics. These notes are written to help introduce you to some of the terms used regularly by mathematicians – specifically the terms we have included in Section 2 of the specification.

Before you launch into reading through what we have written, there are a few things to keep in mind:

1. This guide is designed to be a *brief* overview. It is not designed to be an extensive textbook, but it should be enough to allow you to get a good understanding of the topics in Section 2 of the specification. It should also be sufficient to enable you to understand and tackle the questions we will ask in the admission test.
2. As you read through the guide, make sure you take the time to think through everything very carefully. Many of the ideas set out here are quite subtle and take some time to grasp, so skimming through everything is certainly not enough to ensure you have a good understanding. Rather, you should play around with the ideas as you meet them, and try to come up with your own examples. In other words, read through things actively with a pencil and paper to hand; think carefully about everything, draw your own pictures, write out your own examples, and so on.
3. Another thing you should do is to try to read more widely on the topics set out here. The internet has some good explanations and examples of the ideas we outline and there are some good books available from libraries that might also help.

4. A good way to test if you have understood something is to see if you would be able to explain the ideas to a friend or a class of students. If you get a chance, it is always useful to study the ideas and talk about them with other people – again, maybe with others in your maths classes or with your maths teacher.
5. As you work through this enhanced specification, make sure you are aware that the language used by mathematicians has very precise meanings and these do not always coincide with the way words are used in everyday casual contexts. For instance, if I am told that I may have "jelly **or** cake" for pudding, I would probably assume it meant I could have jelly or cake but not both. However, in mathematics, 'or' means 'one, or the other, or both', so if I were being offered pudding by a mathematician then I could have *both* jelly and cake if I wanted.
6. Throughout, we have tried to explain the ideas in at least two different ways: one using the formal notion of truth tables and one using a more intuitive diagrammatic approach. It is worth making sure you understand both approaches and how they relate to each other.
7. Finally, a small note of caution: in this specification we have tried to take a simple and slightly naïve view of the ideas we are trying to explain rather than one that sets out all the deep subtleties that abound in mathematics and the philosophy of logic/mathematics. We have been as rigorous as necessary to achieve our aims, but you should not take what we have written as the perfect and final word on things and we have deliberately avoided some issues as they would only complicate matters unnecessarily. For instance, some mathematicians or philosophers might take issue with our examples of statements or our notion of truth and so on. For what we aim to achieve these issues are not relevant, but that is not to say they aren't interesting.

SECTION 2

*This section sets out the scope of Paper 2. Paper 2 tests the candidate's ability to think mathematically: the paper will focus on testing the candidate's ability to understand, and construct, mathematical arguments in a variety of contexts. It will draw on the mathematical knowledge outlined in **SECTION 1** in the test specification.*

The Logic of Arguments

Arg1 Understand and be able to use mathematical logic in simple situations:

- The terms **true** and **false**;
- The terms **and**, **or** (meaning **inclusive or**), **not**;
- Statements of the form:

  if A then B
  A if B
  A only if B
  A if and only if B

- The **converse** of a statement;
- The **contrapositive** of a statement;
- The relationship between the truth of a statement and its converse and its contrapositive.

Note: candidates will **not** be expected to recognise or use symbolic notation for any of these terms, nor will they be expected to complete formal truth tables.

Arg2 Understand and use the terms **necessary** and **sufficient**.

Arg3 Understand and use the terms **for all**, **for some** (meaning **for at least one**), and **there exists**.

Arg4 Be able to negate statements that use any of the above terms.

Mathematical Proof

Prf1 Follow a proof of the following types, and in simple cases know how to construct such a proof:

- Direct deductive proof ('Since A, therefore B, therefore C, …, therefore Z, which is what we wanted to prove.');
- Proof by cases (for example, by considering even and odd cases separately);
- Proof by contradiction;
- Disproof by counterexample.

Prf2 Deduce implications from given statements.

Prf3 Make conjectures based on small cases, and then justify these conjectures.

Prf4 Rearrange a sequence of statements into the correct order to give a proof for a statement.

Prf5 Problems requiring a sophisticated chain of reasoning to solve.

Identifying Errors in Proofs

Err1 Identifying errors in purported proofs.

Err2 Be aware of common mathematical errors in purported proofs; for example, claiming 'if $ab = ac$, then $b = c$' or assuming 'if $\sin A = \sin B$, then $A = B$' neither of which are valid deductions.

MATHEMATICAL LOGIC

Statements

The Logic of Arguments

Arg1: The terms **true** and **false**

At the heart of mathematics, and mathematical logic, is the notion of a statement and the relationship between statements. But what can we say about the statements we shall meet in mathematics? We can say that they must be either true or false, but not both. And it does not matter if we cannot actually work out whether a statement is actually true or false so long as it must be one or the other. We shall use this answer to give us a way of understanding roughly what we shall mean by a "statement" in these notes. For present purposes, we shall make do with the following:

> A *statement* is a sentence which is definitely true or definitely false.
> A *statement* can never be both true and false.

The principle that a statement can only be either true or false but not both is known as *the law of the excluded middle*. It is fundamental to all the logic and formal mathematics that you will meet in these notes.

It does not matter if we cannot work out whether a statement is actually true or false so long as it must be one or the other. For instance here is a statement:

> The equation $x^3 + y^2 = 88$ has no integer solutions.

This is clearly either true or false but establishing which is not so easy. Here is a second [rather famous] example of a sentence that is NOT a statement:

> The only barber in a town shaves each and every man who does not shave himself.

This last sentence is not a statement as it is neither true nor false.[1]

Here are some examples:

(1) "It rained yesterday in Auckland, New Zealand." Again, this is a statement, as it is either true or false.
(2) "Go home!" and "What is your name?" These are not statements, as it does not make sense to say that they are true or false.

[1] You do not need to worry at this stage why the sentence is not a statement under our definition; that is, you do not need to worry why it is neither true nor false. If you are interested in the statement and its history look up "Russell's Paradox" or "Barber Paradox".

(3) "If $x = 3$, then $x^2 = 9$." This is certainly true, so it is a statement. We will have more to say about "If … then …" statements later.

(4) "If $x = 3$, then $x^2 = 4$." This is certainly false, so it is a statement. There is no requirement for statements to be true!

(5) "The sum of two odd numbers is an even number." This is certainly true, so it is a statement.

From now on, we will only be working with statements and relationships between statements. We shall try to keep to the convention that we write all our logical statements using italics when they are in words and as bold letters when a statement is indicated by a letter [e.g. **A**, **B**, etc.]. Later we shall use bold for some terms to help us see how they fit into sentences.

A little more on statements

Now we know what we mean by a statement, we shall pause to dig a little deeper into the sorts of statements you might meet and how we discern their truth or falsity. Here is a statement:

24 is divisible by 2

This statement says something that is true and cannot be false, so it is an example of a statement we know to be true just by looking at it. Here is another example:

453653987389875629 is divisible by 987283

This is clearly a statement as it is obviously either true or false. However, whilst it is obvious that it is a statement, it is not so obvious whether it is a true statement or a false statement. To decide that we would need to do some more [tedious] work.

Here is another statement:

The square root of 2 is irrational

This statement is very much like the first statement [*24 is divisible by 2*] in that it is definitely true. However, it is not as obvious as the first statement and some work needs to be done to show why it must be true. Later we shall see how we can set out a proof to show that this statement is true.

And here is an expression that has the potential to be a statement:

The positive integer x is divisible by 2

Here we don't know what x is so we cannot say whether the expression is true or false as it stands; in other words, whether the expression is true or false is conditional on what we are told about x so until we clarify this, we cannot say the expression is a statement according to our definition. We could say it is true or false if we had some more information about x . General expressions like this tend to occur in combination with other statements [and, as you will read later, they need to be quantified in some way – that is, the set of possible x values to which the statement applies must be clearly stated] and then what is often important is what the combination is saying. For instance, the statement:

if *the integer* x *is divisible by 4* **then** *the integer* x *is divisible by 2,*

is definitely true even though each of the expressions that we have combined to make the bigger statement cannot be said to be true or false by themselves.

So we shall meet three sorts of basic statement in what follows:

- those that are obviously true [or obviously false];
- those that are true or false but which need some work to decide which;
- those that are combinations of expressions which are quantified [roughly, that means the range of what the x can be is clearly stated] and are then either true or false [and these will often require some work to decide which they are].

We shall spend a lot of time building new statements out of basic statements.

In what follows we shall tend to learn how various logical rules work by dealing with statements denoted just by letters – such as **A** or **B** or **P** or **Q** - but then we shall apply these rules to statements that have definite truth values. This is a little like learning about quadratic equations by studying various things about $ax^2 + bx + c = 0$ and then applying what you discover to specific examples.

Truth values

In what follows we shall talk a lot about the “truth value” of a statement. By “truth value” we simply mean whether the statement is true or false. For instance, the truth value of the statement *2 is an even number* is “true”, and the truth value of the statement *2 is an odd number* is “false”.

Logically equivalent

We shall often say that two statements are **logically equivalent**. This will mean that the two statements have the same truth values in the same circumstances.[2]

For instance, the following two statements are logically equivalent:

Today is Tuesday

Today is the day after Monday

[2] This is a slightly casual definition but it will suffice for these notes.

Making new statements

Introduction

As we discussed briefly above, mathematics is in part about seeing how the truth or falsity of one statement relates to the truth or falsity of other statements. To help us begin to understand these relationships we shall learn how to build new (compound) statements by formally combining other statements, and we shall learn how the truth or falsity of the combinations depends on the truth or falsity of the statements that we use to build them.

Before we begin to unpack compound statements in detail, here are some examples of formal combinations of statements with the statements written in italics and the formal 'combining terms' set in bold:

21 is divisible by 3 **and** *21 is divisible by 6* [**A and B**]

21 is divisible by 3 **or** *21 is divisible by 6* [**A or B**]

21 is **not** *divisible by 6* [**not B**]

if *21 is divisible by 3* **then** *21 is divisible by 6* [**if A then B**]

21 is divisible by 3 **if** *21 is divisible by 6* [**A if B**]

21 is divisible by 3 **only if** *21 is divisible by 6* [**A only if B**]

21 is divisible by 3 **if and only if** *21 is divisible by 6* [**A if and only if B**]

Exercise A:

1. Decide which of the above combinations are true and which are false. Can you explain your answers?
2. What, if anything, happens to your answers if you replace 21 by x in each of the statements [assume x can be drawn from the set of real numbers]?
3. What happens to your answer to 2 if you change the set of x values to which the statements apply?

The Logic of Arguments

Arg1: The term **not**

In this section we shall look at using **not** with statements. The formal use of **not** in logic is very similar to the everyday use of the term ‘not’ so you will already have a good intuitive grasp of how **not** works.

Formally, if we have a statement **A** then we can construct another statement from it, which we shall write as **not A**, the *negation* of **A**. For instance:

> **A**: *21 is divisible by 3*
> **not A**: **not** [*21 is divisible by 3*]

and we tend to write **not** [*21 is divisible by 3*] as *21 is* **not** *divisible by 3*

You should note that **not** applies only to what occurs immediately after it unless there are brackets: so **not A or B** means **(not A) or B** and, as you will find out later, this is different from **not (A or B)**.

Here we shall learn how to understand the negation of a statement and the relationship between the truth value of a statement and its negation.

Let us start with a simple example of negation:

Example:

Let **A** be the statement:

> *29 is a prime number*

then **not A** is the statement:

> *it is not the case that 29 is a prime number*

which we can write more succinctly as: *29 is **not** a prime number*.

So we have:

> **A**: *29 is a prime number*
>
> **not A**: *29 is* **not** *a prime number*

Here it is obvious that **A** is true and **not A** is false and this is a general property of **not**: it changes true statements into false ones, and it changes false statements to true ones. This rule will always work because, recall, we take statements to be always either true or false [the law of the excluded middle] by definition.

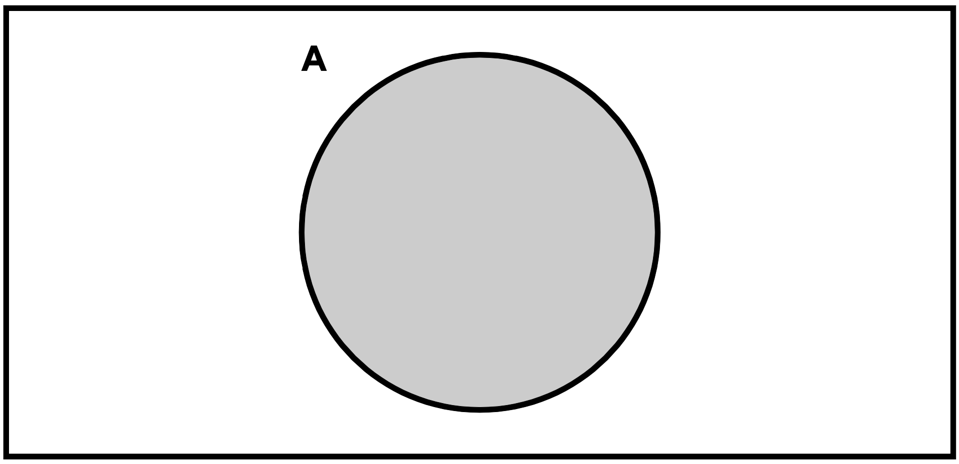
We can display the way **not** works for general statements in one of two ways [in fact, there are other ways but we shall stick to just two ways here]. We can either draw out a ‘truth table’ or we can draw a picture. Let’s start with the truth table:

A	**not A**
T	**F**
F	**T**

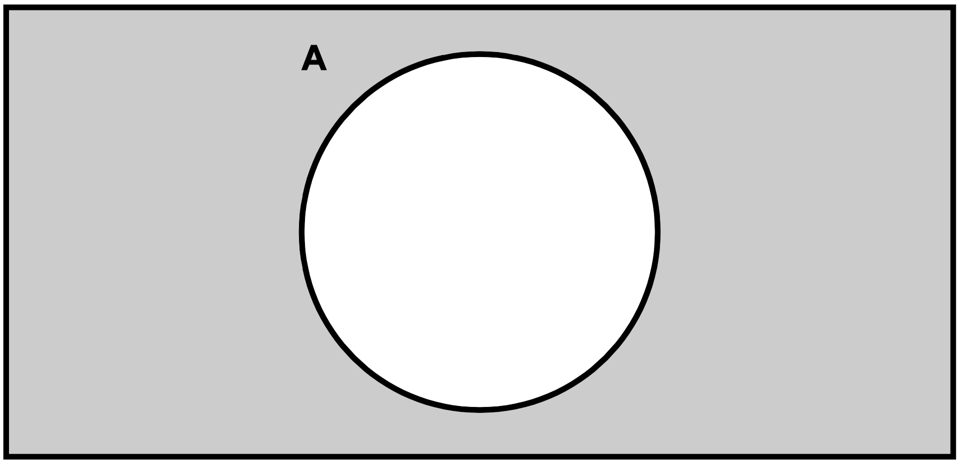
Here **T** is shorthand for “true” and **F** is shorthand for “false”. The first line in the table tells us that whenever **A** is true then **not A** is false, while the second line in the table tells us that whenever **A** is false then **not A** is true.

We can also think about **A** and **not A** using diagrams. The diagrams we will use come from set theory – they are Venn diagrams – and you might have met them when studying probability. The diagrams we use are less formal than truth tables and have a slightly different emphasis – they tend to be useful mostly when talking about general statements, although they can also be useful when thinking through examples with statements that have definite truth values. Here the diagrams are primarily intended to help you think about things.

In the diagrams that follow you should think of the area inside the **A** circle as representing all the cases where **A** is true. And that means you should think of the area outside the circle as representing all the cases when **A** is false; that is, all the cases where **not A** is true. We shall use the convention that each shaded area in a diagram shows where one particular statement is true: **A** is true inside the **A** circle and **not A** is true outside the **A** circle. We shall write what area is shaded [and so what area is true] under each diagram. Here are a couple of examples:



A



not A

If you know some set theory, then you can think of the circle **A** as representing the set where **A** is true, so then **not A** is like the complement of that set. Similarly, if you know about events in probability, then you can think of **A** as an event and **not A** as the complementary event **A'** – the event that **A** does not occur.

Exercise B:

1. If **A** is true, what can you say about **not not A**? What about **not not not A**?
2. Can you work out a general rule for the truth value of **not not not….not A** [m lots of **not**] when **A** is true, and when **A** is false?

The Logic of Arguments

Arg1: The term **and**

In this section we shall look at the logical term **and**. The word “and” appears all over the place in everyday English. However, the use of “and” in logic is very precise – perhaps more so than in colloquial English – so you will need to be a little careful when you use the logical version of “and”.

We begin by setting out a simple example of a compound statement **A and B**:

> **A**: *21 is divisible by 3*
> **B**: *all humans are mammals*

The compound statement is therefore:

> **A and B**: *21 is divisible by 3* **and** *all humans are mammals*

In general, the statement **A and B** is true if each of **A** and **B** are true and it is false if at least one of the statements is false. We could write this up as a table:

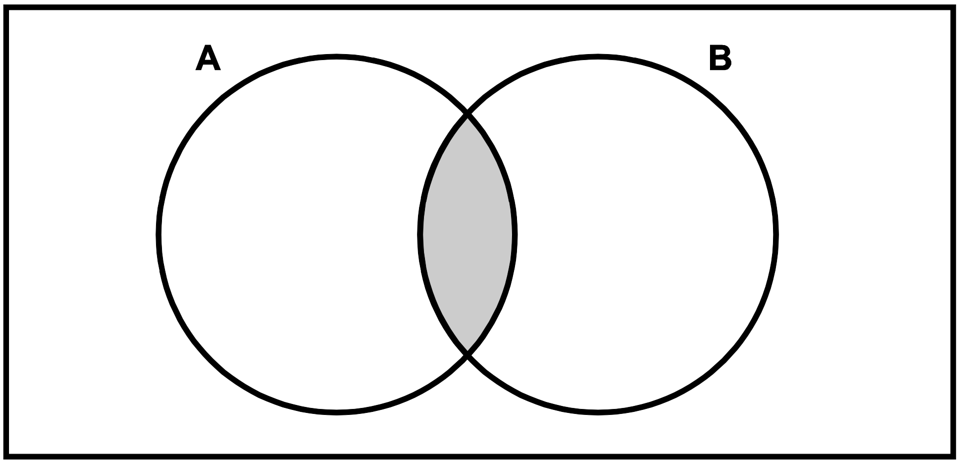
A	B	A and B
T	T	T
T	F	F
F	T	F
F	F	F

Recall, here we have written **T** as shorthand for “true” and **F** for “false”. The table shows that for **A and B** to be true both **A** and **B** must be true.

Let’s look at another example:

> Consider the statement: *the monarch is a woman* **and** *the Prince of Wales is called Charles*. Each of the parts *the monarch is a woman* and the *Prince of Wales is called Charles* is true, so the whole statement is true. (At least this is the case in the UK at the time of writing these notes.)

We can also think of the statement **A and B** using our diagrams. Remember that everything inside the **A** circle is where **A** is true and everything inside the **B** circle is where **B** is true. **A and B** is true when both **A** and **B** are true. So **A and B** is represented by the overlap of the **A** circle and the **B** circle:



A and B

If you know set theory, you can think of **A and B** as being like **A** ∩ **B** [**A** intersect **B**] in diagrams. Similarly, in the language of probability, you can think of **A and B** as the event that both **A** and **B** occur (also written as **A** ∩ **B**).

Combining statements: using or

The Logic of Arguments

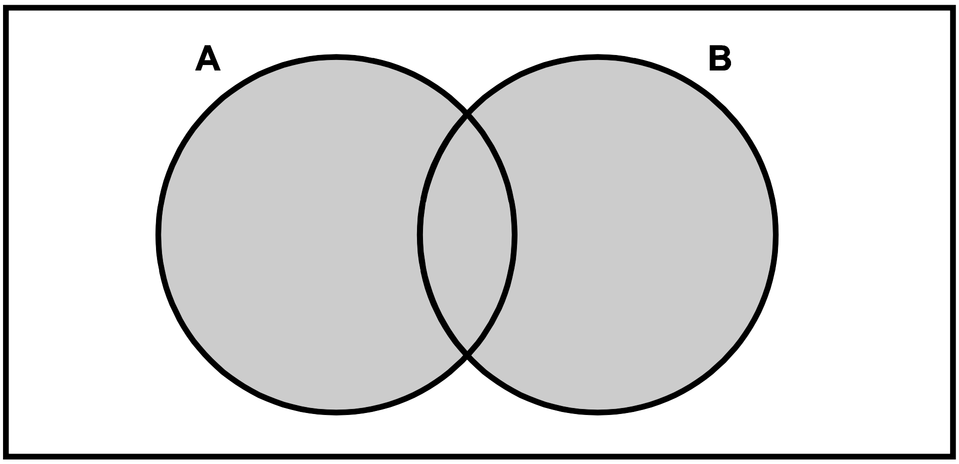
Arg1: The term **or**

The next way we might want to combine two statements is with the word "or". There are two possible meanings of this word. In general usage, "A or B" is often understood to mean "either A is true, or B is true, *but not both*". For instance, in general usage we might hear things like "you can have jam roly-poly or a mille feuille for pudding" and we would usually take that to mean we could have one or the other pudding but not both. This is sometimes called an "exclusive or". However, mathematicians take the word "or" to mean "inclusive or", so that **A or B** means "either **A** is true, or **B** is true, *or both are true*". Over the years, it has been found to be much more convenient to use this version of "or" rather than the "exclusive or". When mathematicians want to mean exclusive or, they are explicit about it, and write "either **A** is true, or **B** is true, but not both". When they just write "or" in a mathematical statement, they *always* mean "inclusive or". This is the meaning of "or" - which we shall write in bold as **or** - that will be used in the admission test.

We can again write a truth table to show this:

A	B	A or B
T	T	T
T	F	T
F	T	T
F	F	F

For example, the statement *the monarch is a woman* **or** *the Prince of Wales is called Charles* is a mathematically true statement, even though it sounds a little strange colloquially.
We can look at **or** using our diagrams. **A or B** is true when we are either inside **A** or inside **B** or inside both. So **A or B** is represented by the shaded region in the following diagram:



A or B

In set theory terms, **A or B** is like **A** ∪ **B** [**A** union **B**] and in probability terms, it is like the event that either **A** or **B** or both occur, also written as **A** ∪ **B**.

Exercise C:

1. Complete the truth table for **A and (B and C)**:

A	B	C	B and C	A and (B and C)
T	T	T	T	T
T	T	F	F	F
T	F	T	F	
T	F	F	F	
F	T	T		
F	T	F		
F	F	T		
F	F	F		

2. Now draw up the truth table for **(A and B) and C**.

   What do you notice? What do you think you can conclude about **A and B and C**?

3. Revisit question 2 above but this time use diagrams to justify your conclusion.

4. Draw up a truth table for each of the following:

A or (B or C)

(A or B) or C

What do you notice? What do you think you can conclude about **A or B or C**?

5. Revisit question 4 but this time use diagrams to justify your conclusions.

6. Draw up truth tables for each of the following:

    A or (B and C)

    (A or B) and (A or C)

    What do you notice? Can you justify your conclusions using diagrams?

    [We say that **or** distributes over **and**.]

7. Can you find an equivalent statement for **A and (B or C)**?

8. How do your results for questions 6 and 7 compare with the arithmetic operations of multiplication and addition?

9. Consider **A and B or C**. Is this statement ambiguous or not? Justify your answer.

10. Draw up truth tables for:

    not (A or B)

    (not A) and (not B)

    What do you notice? Can you justify your conclusions using diagrams?

11. Can you come up with an alternative [logically equivalent] way of writing **not (A and B)**? Justify your alternative statement using both truth tables and diagrams.

Recall, earlier we stated:

"We shall often say that two statements are **logically equivalent**. This will mean that the two statements have the same truth values in the same circumstances."

From the above exercise you should have noticed that **A or (B and C)** and **(A or B) and (A or C)** have the very same truth tables – each expression is true or is false in the same way once you are given the truth values of **A**, **B** and **C**. When two expressions match up in their truth tables in this way, they are logically equivalent. Identity of truth tables is another way to think about logical equivalence.

We can also use logical equivalence to understand **A or B or C** and **A and B and C**. We take it that the statement **A or B or C** is logically equivalent to either **(A or B) or C** or to **A or (B or C)**. We can do the same thing for **A and B and C**. We can justify this as there is no ambiguity when we break **A or B or C** into statements that are of the form "**…or…**" ; that is we can take the statement **A or B or C** and interpret it as either saying **(A or B) or C** or as saying **A or (B or C)**; in both cases we get the same answers for the same truth values of **A**, **B** and **C**. And, we have to work this way – i.e. breaking the statements down using brackets – because we have only defined "or" in situations of the form **… or …**

The same applies for **A and B and C**.

Negating compound statements

The Logic of Arguments

Arg4: Be able to negate statements that use any of the above terms

Negating more complicated statements can be tricky, and truth tables can often help. For example, what is the negation of **A and B**? It is **not (A and B)**, but that use of parentheses looks funny, and it would be tricky to write this as a sentence in English! We can write a truth table for this situation:

A	B	A and B	not (A and B)
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

Which table have we seen earlier which has three trues and one false in the final column? It was the **or** table, so it seems that **not (A and B)** is actually an **or** statement. To get **false** in an **or** statement, we need both parts to be **false**. If we consider **(not A) or (not B)**, both of the parts are false in just the first row, giving the same resulting table:

A	B	A and B	not (A and B)	not A	not B	(not A) or (not B)
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

So **not (A and B)** is the same as **(not A) or (not B)**. And by "the same" we mean they have the same truth values for any given truth values of **A** and **B**; recall that sometimes we say that two statements that have the same truth tables are *logically equivalent*, or just *equivalent*.

As an example, the negation of

x *is even* **and** x *is prime*

is

x *is* **not** *even* **or** x *is* **not** *prime*,

or alternatively, replacing **not** *even* by *odd*:[3]

[3] Here we are assuming x is an integer but we haven't explicitly mentioned it. Later we will look at quantification which deals with this sort of issue formally.

x *is odd* **or** x *is* **not** *prime*.

What about negating **A or B**? Let's look at the truth table:

A	B	A or B	not (A or B)
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

Which table have we seen earlier which has one true (**T**) and three falses (**F**) in the final column? It was the **and** table, so it looks like **not (A or B)** is actually an **and** statement. If we consider **(not A) and (not B)**, we get exactly the same table:

A	B	A or B	not (A or B)	not A	not B	(not A) and (not B)
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

So **not (A or B)** is the same as **(not A) and (not B)**. And, again, by "same" we mean they have the same truth values for any given truth values of **A** and **B**.

Exercise D:

1. Look through a mathematics textbook to find some mathematical statements. What are their negations?
2. Look through some text in English (for example, on a website, in a newspaper or in a book) to find some statements. What are their negations?

STATEMENTS WITH 'IF'

We now turn to look at the part of the specification that involves the word "if".

Language and implication

Before we start to look at how we might understand the formal ideas behind statements that involve "if" in various ways, we shall step sideways to examine how we deal with "if" in everyday English as this will help us when we come to look at the way "if" statements are unpacked in logic.

Here is a collection of statements:

i. *If it is Sunday, then the church bells ring*
ii. *The church bells ring if it is Sunday*
iii. *The church bells ring only if it is Sunday*
iv. *It is Sunday if the church bells ring*
v. *It is Sunday only if the church bells ring*
vi. *The church bells ring if and only if it is Sunday*

Before we start to look at each statement in turn, take a moment to think through what you would understand by each one. Ask yourself: what you would know if the statement is true and it is Sunday; ask yourself what you would know if the statement is true and the bells ring; ask yourself what you can claim about the bells if the statement is true and it is not Sunday; and ask yourself what you can claim about the day if the statement is true and the bells don't ring.

Let's start to look at each statement in turn to see if we can work out what it is telling us and what it is not telling us. In each case we shall assume the statement is true. We start with:

i. *If it is Sunday, then the church bells ring*

First, we ask what does it tell us if we know it is Sunday? It tells us that the church bells will ring. What, then does it tell us about the church bells if it is not Sunday? It tells us nothing; and it tells us nothing because it doesn't tell us about whether the church bells will ring if it is Wednesday or Tuesday and so on. So if we know statement **i** is true and we also know if it is Tuesday then the church bells might ring or they might not ring.

What can we say if we know statement **i** is true and we hear the church bells? Can we say it must be Sunday? The answer is we cannot say it is Sunday. We cannot say it is Sunday as the bells might ring on Tuesday or Wednesday so hearing the bells ring is, according to statement **i**, not enough to tell us what the day is. Finally, what can we determine about the day if the bells do NOT ring? The answer is that we can tell it is NOT Sunday. We can tell it is not Sunday because, if it were Sunday then the bells definitely would ring.

We can summarise these findings as follows:

Statement: **i.** *If it is Sunday, then the church bells ring*

What we know	What we can conclude if **i** is true
It is Sunday	The bells ring
It is not Sunday	Nothing
The bells ring	Nothing
The bells do not ring	It is not Sunday

We can repeat this process for each of the other sentences. We shall summarise the results in a series of tables but take some time to study each one to check it matches any conclusions you have drawn. Some of the examples take some time to think through, particularly statement **iii**:

Statement: **ii.** *The church bells ring if it is Sunday*

What we know	What we can conclude if **ii** is true
It is Sunday	The bells ring
It is not Sunday	Nothing
The bells ring	Nothing
The bells do not ring	It is not Sunday

Here we note that the tables for each of statements **i** and **ii** are identical. The two statements are logically equivalent. That is to say, we take it that *If it is Sunday, then the church bells ring* says the very same thing as *The church bells ring if it is Sunday.* We shall say some more about this below when we start to look at "if" statements formally.

Statement: **iii.** *The church bells ring only if it is Sunday*

What we know	What we can conclude if **iii** is true
It is Sunday	Nothing
It is not Sunday	The church bells do not ring
The bells ring	It is Sunday
The bells do not ring	Nothing

Statement: **iv.** *It is Sunday if the church bells ring*

What we know	What we can conclude if **iv** is true
It is Sunday	Nothing
It is not Sunday	The church bells do not ring
The bells ring	It is Sunday
The bells do not ring	Nothing

Statement: **v.** *It is Sunday only if the church bells ring*

What we know	What we can conclude if **v** is true
It is Sunday	The church bells ring
It is not Sunday	Nothing
The bells ring	Nothing
The bells do not ring	It is not Sunday

Statement: **vi.** *The church bells ring if and only if it is Sunday*

What we know	What we can conclude if **vi** is true
It is Sunday	The church bells ring
It is not Sunday	The church bells do not ring
The bells ring	It is Sunday
The bells do not ring	It is not Sunday

The Logic of Arguments

Arg1: Statements of the form: **if A then B**

So far we have learnt how to use the formal terms **not**, **and** and **or**. In this section we shall look at statements of the form "if…then…". In the previous section we looked informally at this sort of statement when we examined *if it is Sunday, then the church bells ring* and other similar statements. However, we need to be very careful as there aren't definitive rules as to how to interpret these sorts of statements in everyday English, whereas in logic the meaning is precise.

For example, suppose that someone says the statement *if it is raining then I will use my umbrella*. In everyday English, this sentence would be understood with one of the following two meanings:

- If it is raining, I will use my umbrella, while if it is not raining, then I will not use my umbrella.
- If it is raining, I will use my umbrella, while it says nothing at all about what will happen if it is not raining.

When writing mathematical statements, though, we cannot allow such a significant ambiguity. That is why it is important to understand exactly what mathematicians mean when they say **if A then B**.

In logic, the statement **if A then B** means that if **A** is true, then **B** must also be true. But what if **A** is false? What can we say then? In everyday English, different meanings might be understood depending upon the exact sentence and context. But in mathematical logic, this statement has a precise meaning, namely:

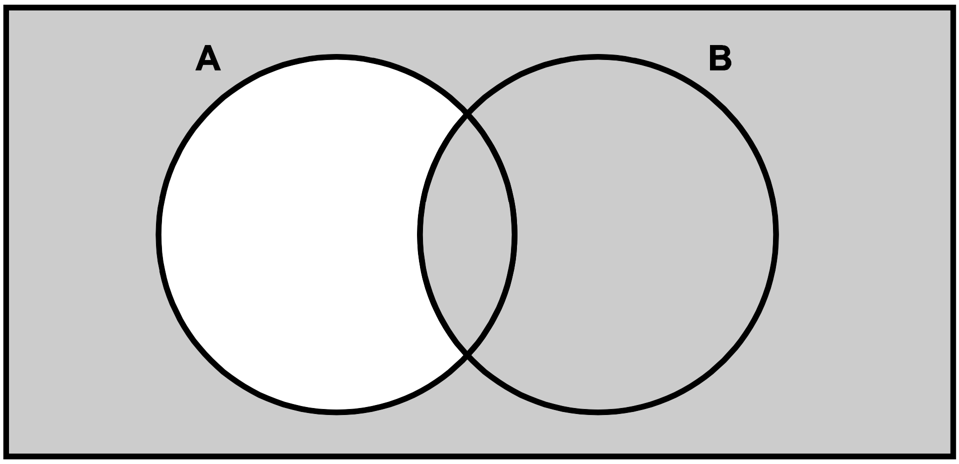
If A is true, then B is true.
If A is false, then B may be either true or false.

Thus the only way that **if A then B** can be false is if **A** is true and **B** is false.

Since **if A then B** is a statement, we can write a truth table for it:

A	B	if A then B
T	T	T
T	F	F
F	T	T
F	F	T

We can look at **if A then B** in a diagram by shading all the areas that make **if A then B** true:



if A then B

Now let us look again at statements **i** and **ii** above:

i. *If it is Sunday, then the church bells ring*
ii. *The church bells ring if it is Sunday*

We can see from the tables we drew up that both statements seem to mean the same thing in everyday English. Using this allows us to introduce another way of writing **if…then…,** statements just using **if**. We do this by defining **if A then B** to be logically equivalent to **B if A**.

Finally, here are some examples of mathematical statements of the form **if A then B**:

- **if** $x = 4$, **then** $x^2 = 8$. This is a false statement, because when $x = 4$ is true, $x^2 = 8$ is false.
- **if** $0 = 1$, **then** $2 + 2 = 5$. This is a true statement, since $0 = 1$ is false. It is true even though $2 + 2 = 5$ is false. This may seem a little strange at first sight!
- **if** a and b are odd integers, **then** $a + b$ is an even integer. This is a true statement, as whenever "a and b are odd integers" is true, so is "$a + b$ is an even integer".
- The standard proofs that $\sqrt{2}$ is irrational begin as follows: "**if** $\sqrt{2}$ is rational, **then** we can write $\sqrt{2} = a/b$, where a and b are integers with $b \neq 0$." This is a true statement, for the only way it could be false is if "$\sqrt{2}$ is rational" is true, but "we can write $\sqrt{2} = a/b$, where a and b are integers with $b \neq 0$" is false.

Exercise E:

1. Notice that the truth table for **if A then B** has three Trues and one False in the final column. Can you guess how **if A then B** might be written in terms of some or all of **and**, **or**, and **not**? Once you have written out your guesses for **if A then B** using **and**, **or** and **not**, can you justify that they have the same truth table as **if A then B**? Can you justify your answer using diagrams?
2. What can you say about the truth of:

    if A then (A or B)
    if A then (A and B)

Equivalent statement for if A then B using not, or, and and

The Logic of Arguments

Arg1: Understand and be able to use mathematical logic in simple situations

In the exercise above we asked you to work out if you could express **if A then B** in an equivalent form using the logical terms **not**, **or** and **and**. In this section we shall explore this a little further as it will be useful for us later.

Let us start by recalling the truth table for **if A then B**:

A	B	if A then B
T	T	T
T	F	F
F	T	T
F	F	T

The final column in this table is similar to the final column of an **or** table. This suggests that **if A then B** is equivalent to some statement involving **or**, but the difficulty is to find the correct **or** statement using statement **A** and statement **B**. We can get a clue from looking at the row in the table where **if A then B** is false: the only situation where an **or** statement can be false is when *both* the statements that make the **or** statement are false. Looking at the row in the table where **if A then B** is false we can see that **A** is true and **B** is false so if we could replace the **T** under **A** with an **F** in this row we would have the correct line in an **or** table. The way to achieve this is to replace **A** by **not A**. This all *suggests* that **if A then B** is equivalent to **not A or B**. Let us construct the truth table for **(not A) or B** and see if it gives us the same table as for **if A then B**:

A	B	not A	not A or B
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

We can see that the two tables do give the same results so we have shown that

if A then B is equivalent to **not A or B**.

Exercise F:

1. Show, using truth tables, that **not (A and not B)** is equivalent to **not A or B**.
2. Show, using truth tables, that **if A then B** is equivalent to **not (A and not B)**.
3. Find [logically]equivalent statements for each of the following:

   a. **if** $x > 1$ **then** $x^2 > 1$
   b. **if** two triangles are similar **then** they have the same angles as each other
   c. **if** a triangle obeys Pythagoras' theorem **then** it has a right angle

The Logic of Arguments

Arg1: Statements of the form: **A only if B**

Above we met statements of the form **if A then B** (or equivalently **B if A**); now we are going to look at statements of the form **A only if B**.

It is hard to untangle the everyday use of the term "only if" from the formal logical use of **only if**. Earlier we asked you to work out what you thought the statement *it is Sunday only if the church bells ring* told you. There you might have noticed that this statement had the same table of conclusions as the statement *if it is Sunday, then the church bells ring*. This motivates what is the case in formal logic: statements of the form **A only if B** are logically equivalent to statements of the form **if A then B**.

Now we shall look at a second example, the true statement

$$\textbf{if } x = 3, \textbf{ then } x^2 = 9$$

This can be written as

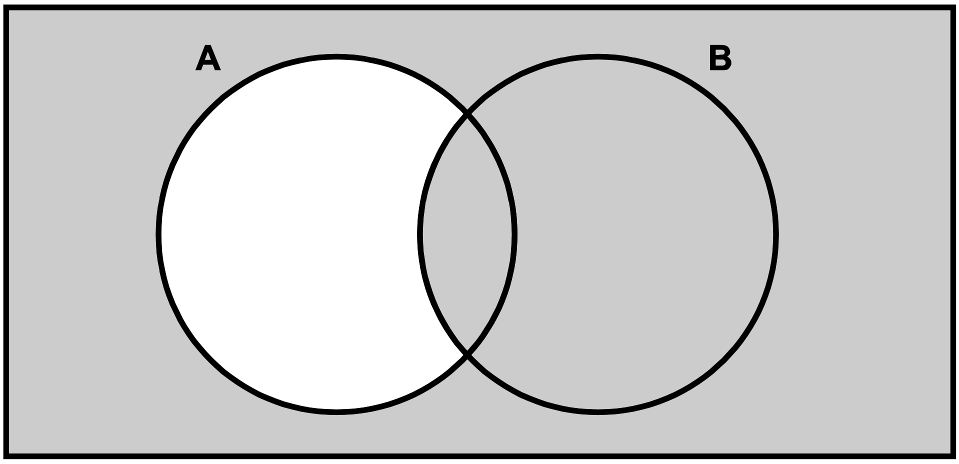
$$x = 3 \textbf{ only if } x^2 = 9$$

This makes some intuitive sense, for if $x^2 \neq 9$, then we cannot have $x = 3$.

We can write out the formal truth table for **A only if B**:

A	B	A only if B
T	T	T
T	F	F
F	T	T
F	F	T

Let’s draw a diagram for **A only if B** to show where it is true:



A only if B

Combining statements: A if and only if B

The Logic of Arguments

Arg1: Statements of the form: **A if and only if B**

Statements of the form **A if and only if B** are very common in mathematics so we shall spend some time unpicking them. First, it is worth noting that **A if and only if B** is often abbreviated to **A iff B** where '**iff**' is usually read as '**if and only if**'. Second, the reason **iff** statements are important is because when they are true they assert that **A** and **B** are really saying the same thing – albeit often in different ways – in that they are both true in all the same circumstances and false in all the same circumstances. In mathematics it is a very useful thing to know when two statements say the same thing in different ways – some might even claim that mathematics is, in essence, about demonstrating that different statements say the same thing. Before we take **iff** statements to pieces and get a feel for how they show two statements are equivalent, let's write out an obvious example:

an integer is even **if and only if** *it is not odd*

When unpicking what **A if and only if B** means it is useful to grasp that it is shorthand for the following:

(A if B) and (A only if B)

And we know from earlier that

A if B is the same as [logically equivalent to] **if B then A**

and

A only if B is the same as [logically equivalent to] **if A then B**

Now we know this, we can use all the rules we have learnt above to construct a truth table to work out when this statement is true and when it is false:

A	**B**	**A if B** **if B then A**	**A only if B** **if A then B**	**(if B then A) and (if A then B)** **A if and only if B** **A iff B**
T	**T**	**T**	**T**	**T**
T	**F**	**T**	**F**	**F**
F	**T**	**F**	**T**	**F**
F	**F**	**T**	**T**	**T**

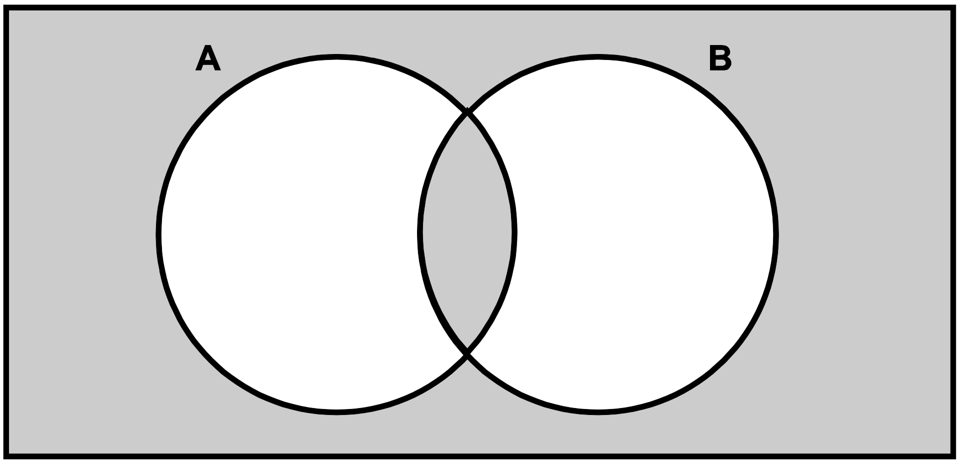
From this we can see that **A iff B** is true when **A** and **B** are both true or when **A** and **B** are both false. That is to say that **A iff B** is true only when **A** and **B** always say the same thing – they are true together and false together. This is why proving **A iff B** is so important for mathematics as it is a way of telling us that two statements that might appear different are really saying the same thing from a mathematical point of view. For instance:

$\lfloor x \rfloor = \lceil x \rceil$ **if and only if** x *is an integer*

where $\lfloor x \rfloor$ is the greatest integer less than or equal to x and $\lceil x \rceil$ is the smallest integer greater than or equal to x. And here is another example:

an integer is divisible by 9 **if and only if** *the sum of its digits is divisible by 9*

How might we illustrate **A iff B** on a diagram? We can approach achieving an answer in two ways: either we can just work it out using diagrams for **if A then B** and for **if B then A** together with the rules for **and**; or we can just shade the areas on a diagram where **A** and **B** are true simultaneously and also where **A** and **B** are false simultaneously. Here is the result:



A if and only if B

Exercise G:

1. Draw the diagrams for **if A then B** [**A only if B**] and for **if B then A** [**A if B**] and then use these two diagrams and the rules for **and** with diagrams to work out the diagram for **A iff B**.

Swapping A and B

The Logic of Arguments

Arg1: Understand and be able to use mathematical logic in simple situations

One thing we have not examined much so far is what happens to each of the logical statements we have examined when we swap **A** with **B**. In this section we shall briefly examine this.

First we look at **A and B**. The question we want to ask is whether **A and B** is the same as **B and A**; and by 'the same' we mean logically equivalent, that **A and B** has the same truth value as **B and A** for any given truth values of **A** and of **B**. The simple answer is 'yes' and this should be obvious from the way we defined **A and B**: our definition was independent of the order of **A and B**.

Exercise H:

1. Examine the truth tables for **A and B** and convince yourself that **A and B** and **B and A** are the same. Look at the diagram we drew for **A and B** and work out what the diagram for **B and A** would look like.

Next we look at **A or B**. The question we want to ask is whether **A or B** is the same as **B or A**; and, again, by 'the same' we mean that **A or B** has the same truth value as **B or A** for any given truth values of **A** and of **B**. The simple answer is 'yes', and again this should be obvious from the way we defined **A or B**: our definition was independent of the order of **A or B**.

What about **if A then B**? Does **if A then B** have the same truth table as **if B then A**? The simple answer is 'no' and we demonstrate this either by looking at the respective truth tables or drawing the respective diagrams. Let's look at the truth table:

A	B	if A then B	if B then A
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

From this we can see that the last two columns are different so the two statements are not the same.

Let's look at **if A then B** and **if B then A** in a little more detail. It's a common error when students are first learning logic to think that one statement is the same [has the same truth profile] as the other. For instance, we might start with the statement:

if $0 < a < b$ **then** $a^2 < b^2$

and it is then tempting to say that this is the same as:

if $a^2 < b^2$ **then** $0 < a < b$

but a little thought shows that they are not equivalent statements. The first is always true no matter what real values of a and b we substitute, whilst the second is false as there are some values of a and b which make it false. For instance, if we set $a = 1$ and $b = -2$ then $a^2 < b^2$ but it's not the case that $0 < a < b$.

Here it is worth pausing for a moment to examine how we have dealt with our example. What do we do when we look at a statement such as **if** $0 < a < b$ **then** $a^2 < b^2$? First, we realise that what we have written, namely **if** $0 < a < b$ **then** $a^2 < b^2$ is shorthand for something a little more precise – we ignored the extra bits above to avoid overloading you with information. What extra information have we ignored here? Well, really the statement **if** $0 < a < b$ **then** $a^2 < b^2$should tell us what values of *a* and *b* it applies to; we ought to write:

for all real values of a *and* b, **if** $0 < a < b$ **then** $a^2 < b^2$

Later we shall say a little more about phrases such as "for all".

With the statement now written out in full, we can return to dealing with the statement: we ask ourselves what happens to the statement when the left-hand side is true and when it is false – do we always find the whole statement is true no matter what allowed values of a and b we substitute into the statement? In the case of **if** $0 < a < b$ **then** $a^2 < b^2$ we see that whenever we have values of a and b that obey [make true] $0 < a < b$ then those same values of a and b must also make the right-hand side – the expression $a^2 < b^2$ – true. So, to say it again, the statement *for all real values of a and b,* **if** $0 < a < b$ **then** $a^2 < b^2$ is always true.

What about the second statement, **if** $a^2 < b^2$ **then** $0 < a < b$? Again we shall take the same approach [we shall assume we have the phrase *for all real values of* a *and* b lurking about]. We ask if there are any values of a and b that make the left-hand side, the $a^2 < b^2$, true and the right-hand side, $0 < a < b$, false. The answer is that there are and we gave such an example above [$a = 1$ and $b = -2$]. So, for the statement **if** $a^2 < b^2$ **then** $0 < a < b$ we can find values of a and b that make the left-hand side true and the right-hand side false. This means that the statement is false.

Now we can return to our main theme: we now consider what happens when we swap **A** with **B** in the statement **A only if B**. Again we can look at the truth table or diagrams to decide whether **A only if B** is the same as **B only if A:**

A	B	A only if B	B only if A
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

It is clear from the truth table that the two statements are not the same – they are not logically equivalent.

Exercise I:

1. Draw diagrams for **A only if B** and for **B only if A** to convince yourself they have different truth profiles.
2. Look at diagrams for all of the following together:

   A or B
   A and B
   if A then B
   A only if B

   Examine the symmetry of the diagrams. What do you notice about the cases that remain unchanged when you swap **A** and **B** and what do you notice about the symmetry of those cases that have different truth tables when you swap **A** and **B**?

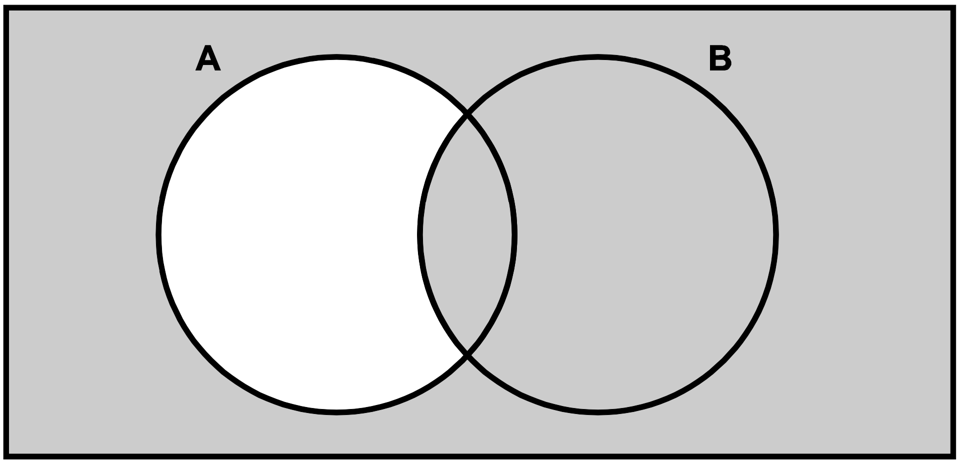
3. Using your answer to 2, what can you say about **A iff B** and **B iff A**, are they the same - do they have the same truth tables for a given **A** and **B**? [Do this before reading the next section.]

Finally, we shall look at the statement **A iff B** and compare it with the statement **B iff A**. Recall that when we first met **A iff B** we said it was a statement that appears a lot in mathematics because it tells us that **A** and **B** are saying the same thing – when **A** is true then **B** is true and vice versa. We can examine whether **A iff B** and **B iff A** say the same thing by looking at truth tables:

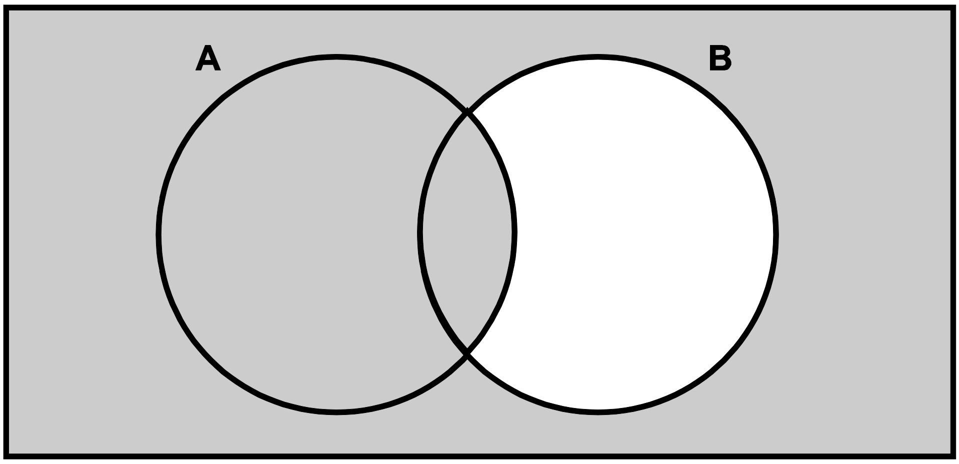
A	B	A iff B	B iff A
T	T	T	T
T	F	F	F
F	T	F	F
F	F	T	T

As an aside, we can also look at how we construct the diagram of **A iff B** from **(if A then B) and (if B then A)**.

First, recall the diagrams of **if A then B [A only if B]** and of **if B then A [A if B]**:

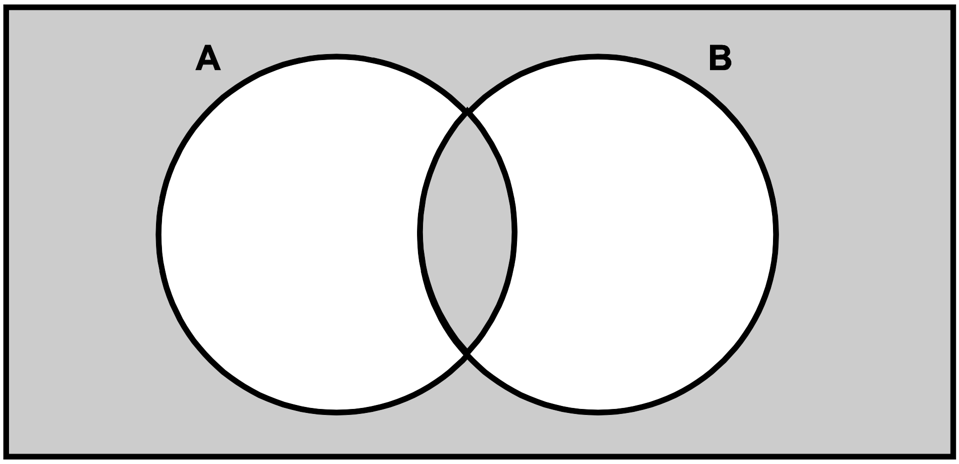


if A then B



if B then A

Then recall that **and** means we shade only those areas that are shaded on both diagrams. When we do this we get:



A if and only if B

And we note that the diagram is symmetric in that it does not matter which circle we label **A** and which we label **B**. Symmetry of diagrams is one way of spotting when the **A** and the **B** can be swapped in a statement without changing the [logical] meaning of the statement.

Summary

Statements

- Can be either true or false but not both
- Can be combined to make "bigger statements"

Same/logically equivalent

- Two statements are the same – logically equivalent – when they have identical truth tables

not A

- Turns false statements into true statements and vice versa
- Applies only to what immediately follows it unless brackets are used
- **not not A** is logically equivalent to **A**
- Truth table:

A	not A
T	F
F	T

A and B

- True only when both **A** and **B** are true, otherwise false
- Symmetry: **A and B** is logically equivalent to **B and A**
- **(A and B) and C** is logically equivalent to **A and (B and C)** and is also logically equivalent to **A and B and C**
- Truth table:

A	B	A and B
T	T	T
T	F	F
F	T	F
F	F	F

A or B

- True when either **A** or **B** or both are true – i.e., true when at least one of the two statements **A**, **B**, is true
- Symmetry: **A or B** is logically equivalent to **B or A**
- **(A or B) or C** is logically equivalent to **A or (B or C)** and is also logically equivalent to **A or B or C**
- Truth table:

A	B	A or B
T	T	T
T	F	T
F	T	T
F	F	F

Negating compound statements:

- **not (A and B)** is logically equivalent to **not A or not B**

- **not (A or B)** is logically equivalent to **not A and not B**

if A then B

- Also written as **B if A** or as **A only if B**
- Not symmetric**: if A then B** is not the same as **if B then A**
- Truth table:

A	**B**	**If A then B**
T	**T**	**T**
T	**F**	**F**
F	**T**	**T**
F	**F**	**T**

A if and only if B

- Also written as **A iff B**
- Symmetric: **A iff B** is logically equivalent to **B iff A**
- Equivalent to **(if B then A) and (if A then B)**
- Equivalent to **(A if B) and (A only if B)**
- Full truth table:

A	**B**	**A if B** **if B then A**	**A only if B** **if A then B**	**(if B then A) and (if A then B)** **A if and only if B** **A iff B**
T	**T**	**T**	**T**	**T**
T	**F**	**T**	**F**	**F**
F	**T**	**F**	**T**	**F**
F	**F**	**T**	**T**	**T**

CONVERSES AND CONTRAPOSITIVES

Converse

The Logic of Arguments

Arg1: The converse of a statement
The relationship between the truth of a statement and its converse

Some of the statements we have met above have what is known as a *converse*. We shall start this section by giving you the converses of a number of statements. Have a look at each example and try to work out how you think we form the converse of a statement. Here are the examples:

Statement	Converse
if *a and b are odd,* ***then*** *ab is odd*	***if*** *ab is odd,* ***then*** *a and b are odd*
if *a and b are even,* ***then*** *ab is even*	***if*** *ab is even,* ***then*** *a and b are even*
if *a is even,* ***then*** a^2 *is even*	***if*** a^2 *is even,* ***then*** *a is even*
if *a is odd,* ***then*** a^2 *is odd*	***if*** a^2 *is odd,* ***then*** *a is odd*
if *a and b are even,* ***then*** $a+b$ *is even*	***if*** $a+b$ *is even,* ***then*** *a and b are even*
if *a and b are odd,* ***then*** $a+b$ *is odd*	***if*** $a+b$ *is odd,* ***then*** *a and b are odd*

Now we shall set out a table of the converses that are relevant to this specification:

Statement	Converse
if A then B	**if B then A**
A only if B	**B only if A**
A if B	**B if A**
A iff B	**B iff A**

From this table you can see that the converse of a statement is constructed by “swapping” **A** with **B**. We have already examined the truth tables of each of the above statements and their converses in earlier sections and we concluded:

- **if A then B** and its converse **if B then A** do NOT say the same thing: they are NOT equivalent statements

- **A only if B** and its converse **B only if A** do NOT say the same thing: they are NOT equivalent statements

- **A if B** and its converse, **B if A** do NOT say the same thing: they are NOT equivalent statements

- **A iff B** and its converse **B iff A** do say the same thing: they are equivalent statements

We can rewrite the table to include logically equivalent statements:

Statement	Converse
if A then B **A only if B** **B if A**	**if B then A** **B only if A** **A if B**
A iff B **B iff A**	**B iff A** **A iff B**

Exercise J:

1. Look back at all the statements we have used as examples so far and write out their converses. How many of the converses are true?
2. What is the converse of the converse of a statement?
3. What is the converse of each of the following:

    a. **if** *two triangles are congruent* **then** *they have the same area*
    b. **if** *two triangles are similar* **then** *they have the same internal angles*
    c. **if** *I am human* **then** *I am mortal* [a classic example from philosophy]
    d. **if** *I am a bachelor* **then** *I am an unmarried man* [another example from philosophy: if you are interested in exploring further, look up analytic and synthetic statements; and if you want to explore more broadly, you could also look at *a priori* and *a posteriori* knowledge, as well as the notion of necessity from a philosophical perspective].

4. Which of the converses you have written out for question 3 are true?

Contraposition

The Logic of Arguments
Arg1: The contrapositive of a statement The relationship between the truth of a statement and its contrapositive

We have learnt that a statement and its converse are not always the same thing. A natural follow-up question to ask is what statements are there that are the same as those we have met – the answer is found in the **contrapositive** of a statement. We shall start by listing the contrapositives that are relevant to this specification, and then we shall examine them in a little more detail:

Statement	Contrapositive
if A then B	**if not B then not A**
A only if B	**not B only if not A**
A iff B	**not B iff not A**

In each of these, **A** and **B** are both swapped and negated, whereas in the converse, they were simply swapped. (There is also a third possibility, called the **inverse** of a statement, where **A** and **B** are both negated, but not swapped. We will not consider inverses further here.)

We examine the truth tables for each of these in turn:

A	B	if A then B	not B	not A	if not B then not A
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

From this we can see that **if A then B** and its contrapositive, **if not B then not A**, are logically equivalent statements.

A	B	A only if B	not B	not A	not B only if not A
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

From this we can see that **A only if B** and its contrapositive, **not B only if not A**, are logically equivalent statements.

Here are a few practical examples of statements and their contrapositives – for each one check you can see why they are equivalent statements and look carefully at how **not** is used in some of the examples:

if $x = 2$ **then** $x^2 = 4$, **if** $x^2 \neq 4$ **then** $x \neq 2$

if $x^2 < 4$ **then** $x < 2$, **if** $x \geq 2$ **then** $x^2 \geq 4$

if *two triangles have the same angles as each other* **then** *they are similar*

if *two triangles are not similar*, **then** *they do not have the same angles as each other*

Exercise K:

1. Look at all the conditional statements that we have set out so far [i.e. all those involving 'if' in one way or another; that is: **if… then…**, **…iff…**, **…only if…**] and work out what their contrapositives say. Can you see, in each case, why the contrapositive is logically equivalent to the original statement?
2. What is the contrapositive of the converse of the statement **if A then B**? Are **if A then B** and the contrapositive of its converse logically equivalent?
3. What is the converse of the contrapositive of the statement **if A then B**? Are **if A then B** and the converse of the contrapositive logically equivalent?
4. What is the contrapositive of **if** *a and b are odd*, **then** *ab is odd*?
5. Why is it a mistake to write the contrapositive of **if** *a and b are odd*, **then** *ab is odd* as **if** *ab is not odd* **then** *a and b are not odd*?

Before you read on, make sure you have completed Exercise K questions 4 and 5 above.

It is important to take care when working out the contrapositive of complicated statements. Consider the statement (about integers) **if** *a and b are odd,* **then** *ab is odd*. This statement is true. Its contrapositive is therefore also true. But what is it? It is very tempting to insert a careless 'not' to produce **if** *ab is not odd* **then** *a and b are not odd,* and from this it is a short step to **if** *ab is even* **then** *a and b are even*. However, this is false [why?], and it is not the contrapositive. The correct form (in English that makes sense and is not strangulated) is:

if *ab is even* **then** *a and b are not both odd*

or

if *ab is even* **then** *at least one of a and b is even*

The reason for this mistake is, of course, that negation is not as simple as it seems – here we need to use **not (A and B)** is logically equivalent to **not A or not B**.

AN OFF-SPECIFICATION ASIDE

Symbols

Not in the specification and not required for the test.

We have now finished looking at the main areas of logic listed in the specification; we have still to look at the notions of necessity, sufficiency and the meaning of some statements such as **for all**, **for some** and **there exists** and we shall return to these below. In this section we shall briefly look at how what we have learnt above is expressed using symbols. We are adding this because it is useful to know how we can write everything we have met using symbols. You should note, however, that the examination will NOT test your ability to use these symbols and these symbols will NOT appear in any of the questions that we set – so, if you want, it is fine to skip this section. Here is a table of common symbols used in formal logic:

What we have met	Alternative/equivalent expression	Formal symbol
A and B	**B and A**	**A ∧ B** **A&B**
A or B	**B or A**	**A ∨ B**
not A		**¬A** [sometimes **~A** or even $\bar{A}$]
if A then B	**A only if B** **B if A**	**A ⟹ B** **B ⟸ A**
if B then A	**B only if A** **A if B**	**B ⟹ A** **A ⟸ B**
A if and only if B	**A iff B** **B iff A**	**A ⟺ B**

Exercise L:

1. Revisit some [or all] of the statements we have met so far in these notes and rewrite them using the symbols above.
2. If you have met electronic circuits, you will probably have met variants on some of these such as **A xor B**, **A nand B** and so on. We will not look at these here but, if you have met them, it would be a useful exercise to examine how they fit with everything we have looked at.

Necessary and Sufficient

The Logic of Arguments

Arg2: Understand and use the terms **necessary** and **sufficient**

The terms **necessary** and **sufficient** turn up a lot in mathematics. You might, for instance, have seen things like:

> *For two triangles to be congruent it is* **sufficient** *that they have two equal sides and the enclosed angle in common.*

Or:

> *For two triangles to be similar it is* **necessary**, *but* **not sufficient**, *that they have an angle in common.*

Or you might have seen parts of questions that say something like:

> *...by considering $(x^2 + gx + h)(x - k)$, or otherwise, show that $g^2 > 4h$ is a* **sufficient** *condition but not a* **necessary** *condition for the inequality*
>
> $$(g - k)^2 > 3(h - gk)$$
>
> *to hold*
>
> [STEP I 2001 question 3]

In this section, we shall explain how mathematicians use the term **necessary** and the term **sufficient**. We need to explain them as they have subtly different features from their everyday uses; the good thing is we have met the notions already, we just didn't refer to them as necessity and sufficiency.

A is sufficient for B means exactly the same as **if A then B**. Usually we think of this as follows: **A is sufficient for B** if we can say that when **A** is true then we are guaranteed that **B** is true as well. And further, we need to note that if **A is sufficient for B** and we find that **A** is actually false, we cannot say whether **B** is true or false - as there might be cases where **B** is true and **A** is false.

The best way to think about **A is sufficient for B** is to think of it as saying that when we know **A** is true then we are guaranteed that **B** is true.

Here is an example:

> *x is an odd natural number greater than 1 and not divisible by any natural number other than 1 and itself is* **sufficient** *for x to be prime.*

This is true because

> **if** *x is an odd natural number greater than 1 and not divisible by any natural number other than 1 and itself* **then** *x is prime.*

is true. It is useful to note here that there is a case where *x is an odd natural number greater than 1 and not divisible by any number other than 1 and itself* is false but *x is prime* is true: i.e., the case $x = 2$. This is fine, though, as it does not make the statement itself false [check the truth table for **if A then B**] and, what is more, it illustrates the point we made above: there we said ‘there might be cases where **B** is true and **A** is false’ and here we have a case of this when $x = 2$.

Now necessity: in simple terms we say that **A is necessary for B** when **if B then A**, or equivalently **A if B**, is true. Usually we think of this as follows: **A is necessary for B** if we can say that when **B** is true then we are guaranteed that **A** is true as well and if **A** is false then **B** must be false as well. And further, as before, we need to note that if **A is necessary for B** and we find that **B** is actually false, we cannot say whether **A** is true or false: there might be cases where **A** is true and **B** is false.

Here is an example of necessity:

> *two triangles having one side of the same length is* ***necessary*** *for the two triangles to be congruent.*

We can see that this necessity condition is quite weak. It tells us something about congruence but not enough to guarantee that two triangles are congruent – two triangles each having one side of the same length is not sufficient to guarantee they are congruent!

We can now look at the term **necessary and sufficient**. From what we have written so far, it should be clear that if we write **A is necessary and sufficient for B**, then we mean **A iff B**. In other words, we mean that when **A** is true **B** is true, and vice versa, and when **A** is false then **B** is false, and vice versa.

Here is an example:

> *two triangles having the same three angles is a* **necessary and sufficient** *condition for the two triangles to be similar.*

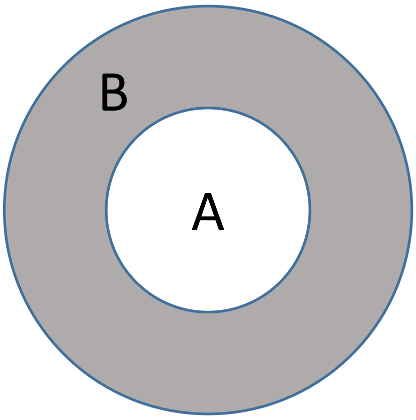
So, to summarise:

- if you are asked for a sufficient condition for **B** to be true then you need to look for a condition that guarantees to make **B** true.

- if you are asked for a necessary condition for **B** to be true then you need to look for something that must be the case for **B** to be true but might not be enough by itself to guarantee that **B** is true.

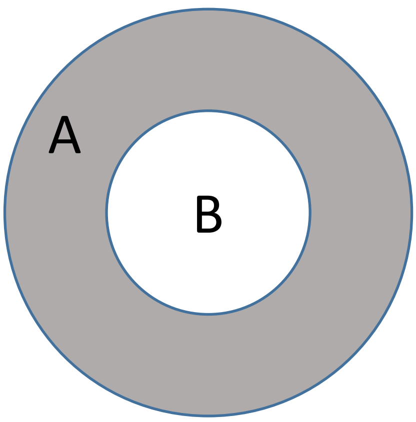
And if you are asked to find **necessary and sufficient** conditions for **B** then you need to look for something that guarantees the truth of **B** in all circumstances: that is, when your condition is true then **B** is true, and vice versa, and when your condition is false then **B** is false, and vice versa.

We can also think about necessity and sufficiency using a diagram. This diagram is slightly different from the diagrams we used earlier – although there are some connections between them – so it is best to look at these diagrams in isolation and treat them as a way of helping you grasp the notions of necessity and sufficiency.



Using the diagram, we can see that **A is sufficient for B**; that is to say, if we are inside the A circle then we must be inside the B circle too. Note that whilst A is sufficient for B, there are cases where we can be inside the B circle but outside the A circle; that is to say that even if A is false there is still the possibility that B is true – you should reconcile this with your formal understanding of **A is sufficient for B**, i.e. with $\mathbf{A} \Longrightarrow \mathbf{B}$

Here is a second diagram with **A** and **B** swapped:



Another way of looking at the diagram is to think about necessity: the diagram shows us that **A is necessary for B**; that is to say, we must be inside the A circle in order to have any chance of being inside the B circle. What is important to note here though is that being inside the A circle is not enough [i.e. is not sufficient] by itself to guarantee we are also inside the B circle. So we need A in order for B to be true but A alone is not enough to guarantee B is true – that is what necessity is all about. Again reconcile this with your formal understanding of **A is necessary for B**, **A $\Leftarrow$ B**

Finally, what happens if **A** is **necessary and sufficient** for **B**? If we look at our diagrams we see that the **A** and the **B** circle need to be covering each other; in other words, the **A** circle and the **B** circle are the very same circle and that means they are really logically equivalent. So **A is necessary and sufficient for B** is another way of saying **A iff B** or even **A $\Leftrightarrow$ B**

Let us finish here by summarising the notions of necessity and sufficiency in tables:

A is sufficient for B	**A only if B**
A is necessary for B	**A if B**
A is necessary and sufficient for B	**A iff B**

A is sufficient for B	**A $\Rightarrow$ B**
A is necessary for B	**A $\Leftarrow$ B**
A is necessary and sufficient for B	**A $\Leftrightarrow$ B**

Quantifiers

The quantifiers for all, for some, and there exists

The Logic of Arguments

Arg3: Understand and use the terms **for all, for some** (meaning **for at least one**), and **there exists.**

Recall earlier, we added some information to a statement to tell us exactly what x values we were considering. There, instead of

if $0 < a < b$ **then** $a^2 < b^2$

we wrote:

for all *real values of* a *and* b, **if** $0 < a < b$ **then** $a^2 < b^2$

In this section we explore phrases such as **for all**, **for some** and **there exists** in more detail.

Consider the following statement:

for all *real* x, $x^2 \geq 0$

This is clearly a true statement but what is important to notice for this section is the phrase "for all". This phrase tells us what our statement applies to – in this case it tells us that the statement, $x^2 \geq 0$, applies to all real numbers. But why do we need to specify what a statement refers to? The reason is that if we don't there might be scope for confusion or ambiguity and mathematics doesn't like confusion or ambiguity. Here is another example:

Consider the statement

x^2 *is an integer*

Now this is sometimes true and sometimes false, for instance it is true when $x = 7$ and it is false when $x = 0.5$. However, if we write:

for all *integers* x, x^2 *is an integer*

it is true, but if we write

for all *real* x, x^2 *is an integer*

it is not true because there are some real x values for which x^2 is not an integer, for instance $x = 0.5$

Two things are important to note here:

1. Mathematicians like to say what their statements apply to and sometimes they do this using phrases like “for all”.
2. Often a statement can be true only in certain situations and mathematicians can use phrases like “for all….” to make it clear what circumstances they are considering.

Sometimes, in place of “for all” we can write “for every” or “for each” so we can take a statement such as

for all *integers* x, x^2 *is an integer*

and rewrite it as:

for every *integer* x, x^2 *is an integer*

or

for each *integer* x, x^2 *is an integer*

Mathematicians also like to assert that something [some mathematical thing, like a number or a function etc.] can be found to make something true, in these cases they tend to use the term “there exists” [usually along with the phrase ‘such that’]. For instance:

there exists *a real* x *such that* $x^2 = 4$

there exists *an* x *such that* x^2 *is an integer*

there exists *a real* x *for which* $x^2 = 4$

Sometimes we might find that a “there exists” statement is actually false; for instance:

there exists *a real integer* x *such that* $x^2 = -4$

Sometimes, in place of **there exists** we can write **for some** or **for at least one** so we can take a statement such as:

there exists *a real* x *such that* $x^2 = 4$

and write it as:

for some *real* x, $x^2 = 4$

or

for at least one *real* x, $x^2 = 4$

Thinking informally about **for all** and **there exists**:

When you see the phrase **for all** x **…** you can think of it as telling you that you can pick ANY x you want from the given set of xs and then the corresponding statement will be true. The phrase is telling you that every value of x makes the statement true.

And when you see the phrase **there exists** *an x such that…* you can think of it as issuing a challenge: you are challenged to FIND an x that makes the statement that the phrase is applied to true. The phrase is telling you that there is at least one x that makes the statement true.

Be aware that **there exists** does not mean that there are any values for which the corresponding statement is false. For example, the statement

there exists *a real x for which $x^2 > -2$*

is true, because $x^2 > -2$ when $x = 0$. It does not matter that $x^2 > -2$ for every real x.

Exercise M:

1. Which of the following are true and which are false?

   i. **for every** *real x, x^2 is rational*
   ii. **there exists** *a real x such that x^2 is rational*
   iii. **for every** *real $x, x^2 > x$*
   iv. **there exists** *a real x such that $x^2 > x$*
   v. **for every** *real $x, x^3 > 0$*
   vi. **there exists** *a real x such that $x^3 > 0$*
   vii. **for every** *real x and $y, x^2 + y^2 > 2xy$*
   viii. **there exists** *real x and y such that $x^2 + y^2 > 2xy$*

Combining the two phrases together

The Logic of Arguments

Arg3: Understand and use the terms **for all, for some** (meaning **for at least one**), and **there exists**

You will often encounter statements in university mathematics that include both the phrase **for all** and the phrase **there exists**. When this happens the order in which they appear is very important. Here are a couple of statements to illustrate how important the order is:

S1: **for all** *positive real* x **there exists** *a real* y *such that* $y^2 = x$

S2: **there exists** *a real* y *such that* **for all** *positive real* x, $y^2 = x$

A little thought will show you that S1 is true but S2 is false. Let us explore why:

S1 is telling us that if we pick any positive x value then we can always find a y value for that x value that obeys the equation $y^2 = x$. In other words, we pick any positive x value first and then look about to see if we can find a y value to go with our chosen x value – and we always can find such a y value so S1 is true. It is useful to note that different choices of x value have different y values associated with them and this is allowed by S1.

S2 is telling us that we can find *one* value of y such that $y^2 = x$ no matter what x value we choose from the positive reals. This is clearly not true. Here the difference is that we are challenged to pick a y value so that our chosen y value then satisfies the test set by the second bit of the statement – we need to test that for our chosen y value it is true that $y^2 = x$ for all positive x values. In other words, to make the statement true we need to find at least one y value such that this one y value obeys all the following [and many more!]: $y^2 = 1,\ y^2 = 2,\ y^2 = 3,\ y^2 = \pi, \ldots$

What we take from these two examples is that the order of the phrases **for all** and **there exists** is important when they occur together and we have to respect the order in which they appear.[4] Only once we have dealt with the first phrase can we then deal with the second phrase in light of what the first phrase has told us.

And a final note: sometimes mathematicians write the phrase "for all real x" (or similar) at the end of a statement instead of at the start, to emphasise the embedded statement, for example:

$$x^2 \geq 0 \text{ **for all** } \textit{real } x$$

[4] There are alternative logics where the restriction on the ordering of the quantifiers is relaxed; look up "independence friendly logic".

This is fine for one occurrence of **for all**, but if it is mixed with **for some** or **there exists** in the same statement, then confusion will result, so it is very unwise to do this.

Exercise N:

1. Identify which of the following are true and which are false:

   i. **for all** *real* x **there exists** *a real* y *such that*: $x > y$
   ii. **for all** *real* x **there exists** *a real* y *such that*: $y > x$
   iii. **for all** *real* y **there exists** *a real* x *such that*: $x > y$
   iv. **for all** *real* y **there exists** *a real* x *such that*: $y > x$
   v. **there exists** *a real* x *such that* **for all** *real* y: $x > y$
   vi. **there exists** *a real* x *such that* **for all** *real* y: $y > x$
   vii. **there exists** *a real* y *such that* **for all** *real* x: $x > y$
   viii. **there exists** *a real* y *such that* **for all** *real* x: $y > x$

Negating for all and there exists

The Logic of Arguments

Arg3: Understand and use the terms **for all, for some** (meaning **for at least one**), and **there exists.**

Earlier we saw what happens when we negated [that is, put **not** in front of] statements such as **A and B**, **A or B** and so on. A natural question to ask is what happens when we use **not** together with **for all** and **there exists**. In this section we shall explore this. Before we begin we should mention again that we tend not just to write "not" in front of statements but translate them into more palatable English: here we shall say "it is not the case that…" in place of **not** by itself.

Let's start by looking at a few examples:

S1: **for all** *real* x, $x^2 > 6$

N1: it is **not** *the case that* **for all** *real* x, $x^2 > 6$

S2: **there exists** *a real* x *such that* $x^2 < 0$

N2: *it is* **not** *the case that* **there exists** *a real* x *such that* $x^2 < 0$

What about the truth of these statements? S1 is false, so N1 is true; S2 is false so N2 is true.

What we want to do is see if we can translate N1 and N2 in some way into a simpler statement. We start with N1: what does N1 say? It says that it's not true that $x^2 > 6$ for all real x values; in other words, it is telling is that there must be some x value for which it is not true that $x^2 > 6$. And in this case we can easily find such an x, for instance $x = 2$. So we now have two equivalent ways of writing out N1:

N1old: *It is* **not** *the case that* **for all** *real* x, $x^2 > 6$

N1new: **there exists** *a real* x *such that* $x^2 > 6$ *is* **not** *the case*

Let's look more carefully at these two versions of N1 to see if we can understand their general structure:

N1old has the structure: **not**-(**for all)** *statement*

N1new has the structure: (**there exists) not**-*statement*

We can actually go further with N1new, by translating "$x^2 > 6$ is **not** the case" into a simpler statement. If $x^2 > 6$ is not the case, then we must have $x^2 \le 6$, so N1 finally becomes

N1newest: **there exists** *a real* x *such that* $x^2 \le 6$

Now let's look at N2: What does N2 say? It says that no matter how hard we look we will never find a real x value that makes $x^2 < 0$ true. In other words, for every real x value the statement $x^2 < 0$ must be false; or, we could say that for every real x value it is not the case that $x^2 < 0$. So we now have two equivalent ways of writing out N2:

N2old: *it is* **not** *the case that* **there exists** *a real* x *such that* $x^2 < 0$

N2new: **for all** *real* x *it is* **not** *the case that* $x^2 < 0$

Let's look more carefully at these two versions of N2 to see if we can understand their general structure:

N2old has the structure: **not**-(**there exists**) *statement*

N2new has the structure: (**for all**) **not**-*statement*

Again, we can simplify our N2new statement one further step to get

N2newest: **for all** *real* x, $x^2 \geq 0$

In summary, we have the following:

not-(**for all** *statement*) is equivalent to (**there exists) not**-*statement*

not-(**there exists** *statement*) is equivalent to (**for all**) **not**-*statement*

Symbols

Not in the specification and not required for the test.

Whilst you are not expected to know, and won't be tested on, the symbols used for the phrases **for all** and **there exists**, it is useful to know what they are and see how the above examples can be translated using these symbols. In this section we shall look, briefly, at the symbolism. In addition, it is worth noting that mathematicians call the phrases **for all** and **there exists** *quantifiers*: **for all** is known as the **universal quantifier** (because it sets the universe of things that you are allowed to consider), and **there exists** is known as the **existential quantifier**.

Now some symbolism:

for all is written as an upside-down A: ∀

there exists is written as a backwards E**:** ∃

These symbols are often combined with set theory and other notation:

∈ to mean "belongs to"

: [a colon] to mean "such that"

¬ to mean **not**

We can now translate some of the statements we looked at in previous sections using this notation:

N1old: *it is* **not** *the case that* **for all** *real* x*,* $x^2 > 6$

N1old translated: $\neg(\forall x \in \mathbb{R}, x^2 > 6)$

N1new: **there exists** *a real* x *such that* $x^2 > 6$ *is* **not** *the case*

N1new translated: $\exists x \in \mathbb{R}: \neg(x^2 > 6)$

N2old: *it is* **not** *the case that* **there exists** *a real* x *such that* $x^2 < 0$

N2old translated: $\neg(\exists x \in \mathbb{R}: (x^2 < 0))$

N2new: **for all** *real* x *it is* **not** *the case that* $x^2 < 0$

N2new translated: $\forall x \in \mathbb{R}, \neg(x^2 < 0)$

Looking at these we can see that in general we have:

¬∀ is the same as ∃¬ ; and ¬∃ is the same as ∀¬

And it is always worth recalling from the discussions we had above that ∀ ... ∃ is *not* generally the same as ∃ ... ∀

And a final note: If a mathematician writes, as mentioned above, something like "$x^2 \geq 0$ **for all** *real* x", it would still be translated into symbols as $\forall x \in \mathbb{R}, x^2 \geq 0$ with the ∀ at the beginning.

PROOF

Mathematical Proof

Prf1 Follow a proof of the following types, and in simple cases know how to construct such a proof

Introduction

Proof is central to mathematics; but what is proof, and why is it so important?

In simple terms a proof is an explanation of why a statement is true. More specifically the proof is a rigorous and convincing explanation of why some statement is true: rigorous in that it must obey mathematical and logical rules throughout; and convincing in that it should be clear enough to convince other mathematicians of its correctness. Proofs can be one line long or they can be very complicated and lengthy, or they can be anything in between. In this section we look at a selection of specific methods of proof; more specifically, we will concentrate on:

- Simple deductive proofs
- Proof by contradiction
- Proof by contrapositive
- Disproof by counterexample

Direct deductive proofs

Mathematical Proof

Prf1 Direct deductive proof ('Since A, therefore B, therefore C,…, therefore Z, which is what we wanted to prove')

Simple deductive proofs tend to ask us to prove **if A then B** type statements. The proof begins with a simple statement **A** that we take to be true and then proceeds through a sequence of smallish, and usually obvious, steps [lots of uses of **if…then…**] each one following from the previous ones. The proof finishes when it reaches the statement **B** which is to be proved. Here is an example:

Let us prove:

$$\textbf{if } x \textit{ is divisible by } 3 \textbf{ then } x^2 \textit{ is exactly divisible by } 9$$

We shall start with $x \textit{ is divisible by } 3$ and keep using **if…then…** statements until we reach $x^2 \textit{ is divisible by } 9$. Each **if…then…** carries the truth of the first statement along with it [because we are using logically valid steps] until we reach the final statement, the conclusion. The conclusion must be true because we will have shown that its truth follows directly from the truth of the first statement in the sequence.

Proof:

$x \text{ is divisible by } 3$

$if\ x \text{ is divisible by } 3 \text{ then } x = 3n, \text{where } n \text{ is an integer}$

$if\ x = 3n \text{ then } x^2 = 9n^2$

$if\ x^2 = 9n^2 \text{ then } x^2 \text{has } 9 \text{ as a factor}$

$if\ x^2 \text{ has } 9 \text{ as a factor then } x^2 \text{ is divisible by } 9 \text{ and so we can conclude that}$

$$\textbf{if } x \text{ is divisible by } 3 \textbf{ then } x^2 \text{ is divisible by } 9$$

We can rewrite this proof more succinctly using some formal notation; remember another way of stating **if A then B** is by saying **A implies B**, and in symbols, this is written as **A ⇒ B**. We can rewrite our proof as follows:

$x \text{ is divisible by } 3$
$\Rightarrow x = 3n, \text{where } n \text{ is an integer}$
$\Rightarrow x^2 = 9n^2$
$\Rightarrow x^2 \text{ has } 9 \text{ as a factor}$
$\Rightarrow x^2 \text{ is divisible by } 9$

What we have done here is combine each line with the next, so rather than writing

$x \text{ is divisible by } 3 \Rightarrow x\ = 3n$
$x = 3n \Rightarrow x^2 = 9n^2$
etc

we avoided repeating ourselves line by line by writing:

$\boldsymbol{x \text{ is divisible by } 3}$
$\Rightarrow x = 3n$
$\Rightarrow x^2 = 9n^2$
etc

We can now look at the general structure of these simple deductive proofs:

If we are asked to prove **A** ⇒ **B** we move from **A** to **B** in a series of small steps:

A ⇒ **P**, **P** ⇒ **Q**, **Q** ⇒ **R**, **R** ⇒ **B**

Which we can write more briefly as:

A ⇒ **P** ⇒ **Q** ⇒ **R** ⇒ **B**

As we mentioned above, this works because we make sure that each step inherits truth from the previous step: remember that if **P** is true and **P** ⇒ **Q** is true then **Q** is true and so on – and we make sure that **P** is true and **P** $\Rightarrow$ **Q** are true by working through a proof of **if A then B** in small steps starting at **A** and ending at **B**.

Mathematical Proof

Prf1 Proof by contradiction

Another type of proof you need to know about is called "proof by contradiction". We shall start this section by setting out a proof that $\sqrt{\mathbf{2}}$ is irrational using this method. We shall then explore how this type of proof works in a little more detail:

To prove: $\sqrt{2}$ is irrational

Proof:
We start by assuming that $\sqrt{2}$ is not irrational, that is we assume that $\sqrt{2}$ is rational. If $\sqrt{2}$ is rational it can be written as a fraction in its lowest terms; that is we can write:

$$\sqrt{2} = \frac{a}{b}$$

where a and b have no factors in common.
Squaring both sides gives us:

$$2 = \frac{a^2}{b^2}$$

which gives:

$$2b^2 = a^2$$

From this we can see that $\boldsymbol{a^2}$ is even.
For $\boldsymbol{a^2}$ to be even, $\boldsymbol{a}$ itself must be even.[5]
And if $\boldsymbol{a}$ is even then $\boldsymbol{a^2}$ is divisible by 4.
If $\boldsymbol{a^2}$ is divisible by 4 then $\boldsymbol{b^2}$ must also be even.
For b^2 to be even, b must be even.
Thus we have a is even and b is even.
This contradicts the assumption that $\frac{a}{b}$ is a fraction in its lowest terms.
This assumption must, therefore, have been false; that is, our assumption that $\sqrt{2}$ is rational must have been false so $\sqrt{2}$ must, in fact, be irrational.

What have we done here? We have taken what we wanted to prove, that $\sqrt{2}$ is irrational, and assumed that it is not true. We have then, through a series of valid logical steps, derived a contradiction. In this case our contradiction is found between our assumption that $\sqrt{2}$ is rational and so can be expressed as a fraction in its lowest terms and the conclusion that both a and b are even. As we have used nothing but

[5] Mathematical proofs vary depending on the audience. You will often have to make some assumptions as to what is well known to your audience. This step could itself be proved but it is a generally accepted statement.

valid logical steps from start to finish, our assumption must have been incorrect. Our assumption was that $\sqrt{2}$ is rational and this must have been incorrect.

We can now set out the general structure of proof by contradiction:

- We are asked to prove some statement **A**.
- We start by assuming **not A** is true.
- We then show that **not A** leads us to two contradictory statements, **B** and **not B**.
- As **B** and **not B** cannot both be true our assumption that **not A** was true must have been an error.
- If **not A** is false, then **A** must be true.

Exercise O:

1. Replace 2 by 9 in the proof that $\sqrt{2}$ is irrational. Why does the proof no longer work?
2. Can you adapt the proof that $\sqrt{2}$ is irrational to show that $\sqrt{p}$ is irrational for all prime p?

Proof by contrapositive

Prf1	Direct deductive proof ('Since A, therefore B, therefore C,..., therefore Z, which is what we wanted to prove')
Arg1	The contrapositive of a statement

If we are asked to prove **if A then B** we can try to prove the contrapositive instead as sometimes this can turn out to be much easier. Remember that the contrapositive of **if A then B** is **if not B then not A** and these statements are logically equivalent – i.e. both expressions say the same thing. Because **if not B then not A** is the very same thing as **if A then B** we can prove the contrapositive of a statement instead of proving the statement itself.

Here is an example:

Prove using the contrapositive:

$$for\ any\ non\ zero\ integer\ x\text{: } \mathbf{if}\ x^3\ is\ odd\ \mathbf{then}\ x\ is\ odd$$

The contrapositive of this statement says

$$for\ any\ non\ zero\ integer\ x\text{: } \mathbf{if}\ x\ is\ not\ odd\ \mathbf{then}\ x^3\ is\ not\ odd$$

And we note that for integers "not odd", means "even" so we need to prove

$$for\ any\ non\ zero\ integer\ x\text{: } \mathbf{if}\ x\ is\ even\ \mathbf{then}\ x^3\ is\ even$$

We can construct this proof:

$If\ \ x\ is\ even\ then\ x =\ 2p\ for\ some\ integer\ p$

$If\ x\ =\ 2p\ then\ x^3\ =\ 8p^3$

$And\ as\ 8p^3\ =\ 2(4p^3)\ and\ as\ p\ is\ an\ integer\ then\ 4p^3\ is\ an\ integer$

$Therefore\ x^3\ =\ 2\ \times\ integer$

$Therefore\ x^3\ is\ even$

$Therefore\ we\ can\ state\ '\mathbf{if}\ x^3\ is\ odd\ \mathbf{then}\ x\ is\ odd'\ is\ true.$

Mathematical Proof

Prf1 Disproof by counterexample

A counterexample to a statement is an example that shows clearly that the statement must be false. We can show that a statement is false merely by finding a counterexample to the statement. This can be useful as it is a quick way of showing a statement is false. It is also good practice to get into the habit of taking statements you meet apart and trying to discern, using examples, why they are true or false. Here is an example:

Example:

Statement: *all prime numbers are odd.*

Counterexample: 2 is a counterexample because 2 is prime but it is even.

Conclusion: the statement *all prime numbers are odd* is false.

What about finding a counterexample to more complex statements? How might we set about finding a counterexample to a statement of the form **if A then B**? First we need to keep in mind that a counterexample is an example where the statement [in this case our statement is **if A then B**] is false, so we need to find an example for **A** and for **B** such that **if A then B** is false: the only way that **if A then B** can be false is if we can find an example of statement **A** that is true and an example of statement **B** that is false. Let us look at an example of this:

Example:

Find a counterexample to the statement: **if** $x < y$ **then** $x^2 < y^2$
To find a counterexample to this statement, we need to find values of x and y that make $x < y$ true but which make $x^2 < y^2$ false. A simple counterexample would be: $x = -2$ and $y = 1$

Exercise P:

1. What would constitute a counterexample to a statement of the form:

    a. **A and B**
    b. **A or B**
    c. **A only if B**
    d. **A iff B**

2. Find a counterexample, if one exists, to each of the following:

   a. *all prime numbers are odd* **and** *greater than 4*
   b. *all prime numbers are odd* **or** *greater than 37*
   c. x *is prime* **if and only if** x *is odd*
   d. x *is odd* **only if** x *is prime*
   e. x *is prime* **only if** x *is odd*
   f. *for all positive odd integers* x*:* x *is prime* ***or*** x *is divisible by some integer* $k < x$

Identifying Errors in Proofs

Err1 Identifying errors in purported proofs.
Err2 Be aware of common mathematical errors in purported proofs; for example, claiming ‘if $ab = ac$, then $b = c$’ or assuming ‘if $\sin A = \sin B$, then $A = B$’ neither of which are valid deductions.

There are lots of pitfalls in setting out proofs and you should start to collect a set of examples of where proofs can go wrong and look out for these sorts of errors and misunderstandings in your own work and in proofs that you are given to study. In this section we shall look at a few examples of the sorts of mistakes and errors that can occur in proofs; but, be warned, this is not an exhaustive list and there are many errors that mathematicians can make when setting out proofs.

Square roots and squaring equations:

In this specification we shall take it that $\sqrt{x}$ means the *positive* number y such that $y^2 = x$; this is standard in mathematics. Generally we need to be careful with equations when we square them. We need to be careful in case we generate extra solutions to the equation. Here are two examples:

Example 1:
Given $x = \sqrt{25}$ [recall this means $x = +5$]
Square both sides: $x^2 = 25$
Find all values of x which make $x^2 = 25$ true: $x = \pm 5$
So we have generated an extra solution, namely $x = -5$ which we didn’t have to start with.

Example 2:
Find x given $x + 1 = 4$
Square both sides: $(x + 1)^2 = 16$
Giving: $x^2 + 2x + 1 = 16$
or, $x^2 + 2x - 15 = 0$
Factorising: $(x + 5)(x - 3)$
Giving solutions: $x = -5$ or $x = +3$
Here we can see that by squaring the original equation we have generated an extra solution, namely $x = -5$

Exercise Q:

1. Solve : $\sqrt{2x+3} + \sqrt{x+1} = \sqrt{7x+4}$

Dealing with inequality signs:

When students first meet inequality signs, they naturally assume they behave the same way as equals signs: they don't. For an equals sign, the general rule that students go by is that "whatever you do to one side of the equals sign you must do to the other" and this is usually fine for equals signs. However, if you use this rule within inequality signs then it might be the case that the inequality is no longer preserved. Here are some pitfalls that you need to watch out for:

Squaring both sides:

$-5 \ < \ 4$ is correct but on squaring we obtain the false $25 < 16$

Multiplying both sides by a negative number:

$1 < 2$ is true, but on multiplying by -1 we obtain the false $-1 < -2$

Taking some function of both sides:

$\frac{\pi}{4} < \frac{\pi}{3}$ is true but on taking the cosine of both sides we obtain the false $\cos\frac{\pi}{4} < \cos\frac{\pi}{3}$

Exercise R:

1. Given $x < y$ what positive integer values of n make $x^n < y^n$ true?
2. What general characteristics would the function f need to have to make $f(x) < f(y)$ given $x < y$? [Discuss this with others in your class.]
3. Starting with $\frac{x+2}{2x+7} < 5$, is it then valid to deduce $x + 2 < 5(2x + 7)$? Justify your answer. [Discuss this with others in your class.]

Dividing or multiplying by zero:

Dividing both sides of an expression by a second expression that is equal to zero can cause problems. Generally, we cannot divide by zero as it can generate nonsense. For instance, we know $7 \times 0 = 5 \times 0$ but we cannot divide both sides by 0 to give $7 = 5$. This issue extends to examples that contain algebra. Here is a classic proof that commits this error [can you spot exactly where the error occurs?]:

Let x and y be non-zero numbers such that $x \ = \ y$
Then we can write $x^2 \ = \ xy$
Subtract y^2 from both sides: $x^2 \ - \ y^2 \ = \ xy \ - \ y^2$
So $(x + y)(x - y) = y(x - y)$
Dividing by $(x - y)$: $x \ + \ y \ = \ y$
As $x \ = \ y$, we have: $2y \ = \ y$
Then dividing by the non-zero number y: $2 \ = \ 1$
Subtracting 1 from both sides: $1 = \ 0$
Therefore $1 = 0$

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