STEP III, 2024, Q7 MS

7. (i) Each term of $f(n) > 0$ so their sum is too.

E1

$\frac{1}{(n+1)(n+2)\ldots(n+r)} < \frac{1}{(n+1)^r}$ so $f(n) = \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \cdots < \frac{1}{n+1} + \frac{1}{(n+1)^2} + \cdots = \frac{1/n+1}{1-1/n+1} = \frac{1}{n}$

M1 A1(3)

Thus $0 < f(n) < \frac{1}{n}$

(ii) $\frac{1}{n+1} - \frac{1}{(n+1)(n+2)} > 0$, $\frac{1}{(n+1)(n+2)(n+3)} - \frac{1}{(n+1)(n+2)(n+3)(n+4)} > 0$, etc so $g(n) > 0$

M1 A1

Also, $\frac{1}{(n+1)(n+2)} - \frac{1}{(n+1)(n+2)(n+3)} > 0$, $\frac{1}{(n+1)(n+2)(n+3)(n+4)} - \frac{1}{(n+1)(n+2)(n+3)(n+4)(n+5)} > 0$, etc

so $g(n) = \frac{1}{n+1}$ −a sum of positive terms $< \frac{1}{n+1}$ **M1 A1 (4)**

Thus $0 < g(n) < \frac{1}{n+1}$

(iii)

$$(2n)!\,e - f(2n)$$

$$= (2n)!\left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots\right) - \frac{1}{2n+1} - \frac{1}{(2n+1)(2n+2)} - \frac{1}{(2n+1)(2n+2)(2n+3)} - \cdots$$

M1 A1

$$= (2n)!\left(1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{(2n)!}\right)$$

$$+ \frac{(2n)!}{(2n+1)!} + \frac{(2n)!}{(2n+2)!} + \cdots - \frac{1}{2n+1} - \frac{1}{(2n+1)(2n+2)} - \frac{1}{(2n+1)(2n+2)(2n+3)} - \cdots$$

$$= (2n)!\left(1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{(2n)!}\right)$$

which is an integer. **M1 A1(4)**

$$\frac{(2n)!}{e} + g(2n) = (2n)!\,e^{-1} + g(2n)$$

$$= (2n)!\left(1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots\right) + \frac{1}{2n+1} - \frac{1}{(2n+1)(2n+2)} + \frac{1}{(2n+1)(2n+2)(2n+3)} - \cdots$$

M1 A1

$$= (2n)!\left(1 - 1 + \frac{1}{2!} + \cdots + \frac{1}{(2n)!}\right)$$

$$-\frac{(2n)!}{(2n+1)!} + \frac{(2n)!}{(2n+2)!} - \cdots + \frac{1}{2n+1} - \frac{1}{(2n+1)(2n+2)} + \frac{1}{(2n+1)(2n+2)(2n+3)} - \cdots$$

$$= (2n)!\left(1 - 1 + \frac{1}{2!} + \cdots + \frac{1}{(2n)!}\right)$$

M1 A1 (4)

which is an integer.

(iv) $q\big((2n)!\,e - f(2n)\big)$ is an integer as is $p\left(\frac{(2n)!}{e} + g(2n)\right)$

Thus $\left(p\left(\frac{(2n)!}{e} + g(2n)\right)\right) - \big(q\big((2n)!\,e - f(2n)\big)\big)$ is an integer.

$$\left(p\left(\frac{(2n)!}{e} + g(2n)\right)\right) - \big(q\big((2n)!\,e - f(2n)\big)\big) = (2n)!\left(\frac{p}{e} - qe\right) + pg(2n) + qf(2n)$$

M1

$= pg(2n) + qf(2n)$

so $pg(2n) + qf(2n)$ is an integer as required. **A1(2)**

(v) As (iv) is true for all positive integers n, it must be true for $n = \max(p, q)$

By (ii) $\quad pg(2n) < \frac{p}{2n+1} \le \frac{n}{2n+1} < \frac{1}{2}$

By (i) $\quad qf(2n) < \frac{q}{2n} \le \frac{n}{2n} = \frac{1}{2}$

M1

Therefore, $\quad pg(2n) + qf(2n) < \frac{1}{2} + \frac{1}{2} = 1$

and trivially by (i) and (ii)

$$pg(2n) + qf(2n) > 0$$

A1

This means that $pg(2n) + qf(2n)$ cannot be an integer which contradicts the result of (iv) and hence there are no integers such that $\frac{p}{e} = qe$, that is such that $\frac{p}{q} = e^2$ and so e^2 is irrational. **E1 (3)**