STEP III, 2024, Q6 MS

6. (i) (a)

$$\frac{dx}{dt} = -x + 3y + u$$

$$\frac{dy}{dt} = x + y + u$$

$$\frac{dx}{dt} - \frac{dy}{dt} = \frac{d(x-y)}{dt} = -2(x-y)$$

M1

$$x - y = Ae^{-2t}$$

A1

If $x = y = 0$ at some time $t > 0$, then $A = 0$, **A1**

so considering $t = 0$, $x_0 - y_0 = 0$ which gives the required result. **E1 (4)**

(b) If $x_0 = y_0$, then at $t = 0$, $x - y = 0$ so $A = 0$ and hence $x = y$ for all t

E1

Thus

$$\frac{dx}{dt} = 2x + u$$

$$\frac{dx}{dt} - 2x = u$$

$$e^{-2t}\frac{dx}{dt} - 2e^{-2t}x = e^{-2t}u$$

$$e^{-2t}x = -\frac{1}{2}e^{-2t}u + c$$

$$x = -\frac{1}{2}u + ce^{2t}$$

M1 A1

$t = 0$, $x = x_0$ so $x_0 = -\frac{1}{2}u + c$ and we want $x = 0$ when $t = T$

so

$$0 = -\frac{1}{2}u + ce^{2T}$$

Thus $c = \frac{1}{2}ue^{-2T}$, $x_0 = -\frac{1}{2}u + \frac{1}{2}ue^{-2T}$

and hence,

$$u = \frac{2x_0e^{2T}}{1 - e^{2T}}$$

dM1 A1 (5)

(ii) (a)

$$\frac{dx}{dt} - 2\frac{dy}{dt} + \frac{dz}{dt} = \frac{d(x-2y+z)}{dt} = -(x-2y+z)$$

Thus

$$x - 2y + z = Ae^{-t}$$

M1 A1

If $x = y = z = 0$ at some time $t > 0$, then $A = 0$, so considering $t = 0$, $x_0 - 2y_0 + z_0 = 0$ which gives the required result. **E1 (3)**

(b) we know from (a) that if $x = y = z = 0$ at some time $t > 0$, then $A = 0$, and so

$x - 2y + z = 0$ or $2y = x + z$ **E1**

Thus

$$\frac{dx}{dt} = 2x - 3z + u$$

and

$$\frac{dz}{dt} = -z + u$$

So

$$\frac{dx}{dt} - \frac{dz}{dt} = \frac{d}{dt}(x - z) = 2(x - z)$$

and so

$$x - z = Be^{2t}$$

M1 A1

But as $x = z = 0$ at some time $t > 0$, $B = 0$ and so $x = z$ for all t

and thus $x = y = z$ for all t

Hence

$$x_0 = y_0 = z_0$$

E1 (4)

(c)

Given

$$x_0 = y_0 = z_0$$

we know that (a) and (b) apply (as similarly in (i)), so

$$\frac{dz}{dt} = -z + u$$

M1

Thus

$$z = u + ce^{-t}$$

A1

$t = 0$, $z = z_0$ so $z_0 = u + c$ and $0 = u + ce^{-T}$

dM1

$$c = -ue^{T}$$

$$u = \frac{z_0}{1 - e^{T}}$$

A1 (4)