

## STEP III, 2024, Q6 MS

6. (i) (a)

$$\frac{dx}{dt} = -x + 3y + u$$

$$\frac{dy}{dt} = x + y + u$$

$$\frac{dx}{dt} - \frac{dy}{dt} = \frac{d(x-y)}{dt} = -2(x-y)$$

**M1**

$$x - y = Ae^{-2t}$$

**A1**

If  $x = y = 0$  at some time  $t > 0$ , then  $A = 0$ , **A1**

so considering  $t = 0$ ,  $x_0 - y_0 = 0$  which gives the required result. **E1 (4)**

(b) If  $x_0 = y_0$ , then at  $t = 0$ ,  $x - y = 0$  so  $A = 0$  and hence  $x = y$  for all  $t$

**E1**

Thus

$$\frac{dx}{dt} = 2x + u$$

$$\frac{dx}{dt} - 2x = u$$

$$e^{-2t} \frac{dx}{dt} - 2e^{-2t}x = e^{-2t}u$$

$$e^{-2t}x = -\frac{1}{2}e^{-2t}u + c$$

$$x = -\frac{1}{2}u + ce^{2t}$$

**M1 A1**

$t = 0$ ,  $x = x_0$  so  $x_0 = -\frac{1}{2}u + c$  and we want  $x = 0$  when  $t = T$

so

$$0 = -\frac{1}{2}u + ce^{2T}$$

Thus  $c = \frac{1}{2}ue^{-2T}$ ,  $x_0 = -\frac{1}{2}u + \frac{1}{2}ue^{-2T}$

and hence,

$$u = \frac{2x_0e^{2T}}{1 - e^{2T}}$$

**dM1 A1 (5)**



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(ii) (a)

$$\frac{dx}{dt} - 2\frac{dy}{dt} + \frac{dz}{dt} = \frac{d(x - 2y + z)}{dt} = -(x - 2y + z)$$

Thus

$$x - 2y + z = Ae^{-t}$$

**M1 A1**

If  $x = y = z = 0$  at some time  $t > 0$ , then  $A = 0$ , so considering  $t = 0$ ,  $x_0 - 2y_0 + z_0 = 0$  which gives the required result. **E1 (3)**

(b) we know from (a) that if  $x = y = z = 0$  at some time  $t > 0$ , then  $A = 0$ , and so

$$x - 2y + z = 0 \quad \text{or} \quad 2y = x + z \quad \mathbf{E1}$$

Thus

$$\frac{dx}{dt} = 2x - 3z + u$$

and

$$\frac{dz}{dt} = -z + u$$

So

$$\frac{dx}{dt} - \frac{dz}{dt} = \frac{d}{dt}(x - z) = 2(x - z)$$

and so

$$x - z = Be^{2t}$$

**M1 A1**

But as  $x = z = 0$  at some time  $t > 0$ ,  $B = 0$  and so  $x = z$  for all  $t$

and thus  $x = y = z$  for all  $t$

Hence

$$x_0 = y_0 = z_0$$

**E1 (4)**



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(c)

Given

$$x_0 = y_0 = z_0$$

we know that (a) and (b) apply (as similarly in (i)), so

$$\frac{dz}{dt} = -z + u$$

**M1**

Thus

$$z = u + ce^{-t}$$

**A1**

$t = 0$ ,  $z = z_0$  so  $z_0 = u + c$  and  $0 = u + ce^{-T}$

**dM1**

$$c = -ue^T$$

$$u = \frac{z_0}{1 - e^T}$$

**A1 (4)**



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