STEP III, 2024, Q6

6 **(i)** A particle moves in two-dimensional space. Its position is given by coordinates (x, y) which satisfy

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -x + 3y + u$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = x + y + u$$

where t is the time and u is a function of time. At time $t = 0$, the particle has position (x_0, y_0).

(a) By considering $\dfrac{\mathrm{d}x}{\mathrm{d}t} - \dfrac{\mathrm{d}y}{\mathrm{d}t}$, show that if the particle is at the origin $(0, 0)$ at some time $t > 0$, then it is necessary that $x_0 = y_0$.

(b) Given that $x_0 = y_0$, find a constant value of u that ensures that the particle is at the origin at a time $t = T$, where $T > 0$.

(ii) A particle whose position in three-dimensional space is given by co-ordinates (x, y, z) moves with time t such that

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 4y - 5z + u$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = x - 2z + u$$
$$\frac{\mathrm{d}z}{\mathrm{d}t} = x - 2y + u$$

where u is a function of time. At time $t = 0$, the particle has position (x_0, y_0, z_0).

(a) Show that, if the particle is at the origin $(0, 0, 0)$ at some time $t > 0$, it is necessary that y_0 is the mean of x_0 and z_0.

(b) Show further that, if the particle is at the origin $(0, 0, 0)$ at some time $t > 0$, it is necessary that $x_0 = y_0 = z_0$.

(c) Given that $x_0 = y_0 = z_0$, find a constant value of u that ensures that the particle is at the origin at a time $t = T$, where $T > 0$.