STEP III, 2024, Q3

3 *Throughout this question, consider only $x > 0$.*

(i) Let

$$g(x) = \ln\left(1 + \frac{1}{x}\right) - \frac{x + c}{x(x+1)}$$

where $c \geqslant 0$.

(a) Show that $y = g(x)$ has positive gradient for all $x > 0$ when $c \geqslant \frac{1}{2}$.

(b) Find the values of x for which $y = g(x)$ has negative gradient when $0 \leqslant c < \frac{1}{2}$.

(ii) It is given that, for all $c > 0$, $g(x) \to -\infty$ as $x \to 0$.

Sketch, for $x > 0$, the graphs of

$$y = g(x)$$

in the cases

(a) $c = \frac{3}{4}$,

(b) $c = \frac{1}{4}$.

(iii) The function f is defined as

$$f(x) = \left(1 + \frac{1}{x}\right)^{x+c}.$$

Show that, for $x > 0$,

(a) f is a decreasing function when $c \geqslant \frac{1}{2}$;

(b) f has a turning point when $0 < c < \frac{1}{2}$;

(c) f is an increasing function when $c = 0$.