STEP III, 2024, Q1 MS

1. (i)

$$\frac{1}{r+1}-\frac{1}{r}+\frac{1}{r^2}=\frac{1}{r^2(r+1)}[r^2-r(r+1)+(r+1)]$$

$$=\frac{1}{r^2(r+1)}[r^2-r^2-r+r+1]=\frac{1}{r^2(r+1)}$$

***B1**

Thus

$$\sum_{r=1}^{N}\frac{1}{r^2(r+1)}=\sum_{r=1}^{N}\frac{1}{r+1}-\sum_{r=1}^{N}\frac{1}{r}+\sum_{r=1}^{N}\frac{1}{r^2}$$

$$=\sum_{r=2}^{N+1}\frac{1}{r}-\sum_{r=1}^{N}\frac{1}{r}+\sum_{r=1}^{N}\frac{1}{r^2}$$

M1

$$=\frac{1}{N+1}-1+\sum_{r=1}^{N}\frac{1}{r^2}$$

as required. ***A1**

So, as $N\to\infty$,

$$LHS\to\sum_{r=1}^{\infty}\frac{1}{r^2(r+1)},\ \frac{1}{N+1}\to 0\text{, and }\sum_{r=1}^{N}\frac{1}{r^2}\to\frac{1}{6}\pi^2$$

and hence

$$\sum_{r=1}^{\infty}\frac{1}{r^2(r+1)}=\frac{1}{6}\pi^2-1$$

***B1 (4)**

(ii)

$$\frac{1}{r^2(r+1)(r+2)}=\frac{A}{r}+\frac{B}{r^2}+\frac{C}{r+1}+\frac{D}{r+2}$$

M1

$$1=Ar(r+1)(r+2)+B(r+1)(r+2)+Cr^2(r+2)+Dr^2(r+1)$$

$$r=0\quad 1=2B\quad B=\frac{1}{2}$$

$$r=-1\quad 1=C$$

$$r=-2\quad 1=-4D\quad D=-\frac{1}{4}$$

$$r^3\text{ terms}\quad 0=A+C+D\quad A=-\frac{3}{4}$$

M1 A1 (3)

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Thus

$$\sum_{r=1}^{N}\frac{1}{r^2(r+1)(r+2)}=-\frac{3}{4}\sum_{r=1}^{N}\frac{1}{r}+\frac{1}{2}\sum_{r=1}^{N}\frac{1}{r^2}+\sum_{r=1}^{N}\frac{1}{r+1}-\frac{1}{4}\sum_{r=1}^{N}\frac{1}{r+2}$$

$$=-\frac{3}{4}\sum_{r=1}^{N}\frac{1}{r}+\frac{1}{2}S_N+\sum_{r=2}^{N+1}\frac{1}{r}-\frac{1}{4}\sum_{r=3}^{N+2}\frac{1}{r}$$

M1

$$=-\frac{3}{4}-\frac{3}{8}+\frac{1}{2}S_N+\frac{1}{2}+\frac{1}{N+1}-\frac{1}{4}\frac{1}{N+1}-\frac{1}{4}\frac{1}{N+2}$$

dM1 A1ft

So

$$\sum_{r=1}^{N}\frac{1}{r^2(r+1)(r+2)}=\frac{1}{2}S_N+\frac{3}{4}\frac{1}{N+1}-\frac{1}{4}\frac{1}{N+2}-\frac{5}{8}$$

and taking limits as $N\to\infty$

M1

$$\sum_{r=1}^{\infty}\frac{1}{r^2(r+1)(r+2)}=\frac{1}{2}\times\frac{1}{6}\pi^2-\frac{5}{8}=\frac{1}{12}\pi^2-\frac{5}{8}$$

A1 (5)

(iii)

$$\frac{1}{r^2(r+1)^2}=\frac{A}{r}+\frac{B}{r^2}+\frac{C}{r+1}+\frac{D}{(r+1)^2}$$

M1

$$1=Ar(r+1)^2+B(r+1)^2+Cr^2(r+1)+Dr^2$$

$$r=0\quad 1=B$$

$$r=-1\quad 1=D$$

$$r^3\ terms\quad 0=A+C$$

$$r^2\ terms\quad 0=2A+B+C+D$$

M1

$$-2=2A+C$$

$$A=-2\quad C=2$$

Thus $$\frac{1}{r^2(r+1)^2}=\frac{-2}{r}+\frac{1}{r^2}+\frac{2}{r+1}+\frac{1}{(r+1)^2}$$

A1 (3)

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$$\sum_{r=1}^{N}\frac{1}{r^2(r+1)^2}=\sum_{r=1}^{N}\frac{-2}{r}+\sum_{r=1}^{N}\frac{1}{r^2}+\sum_{r=1}^{N}\frac{2}{(r+1)}+\sum_{r=1}^{N}\frac{1}{(r+1)^2}$$

$$=\sum_{r=1}^{N}\frac{-2}{r}+\sum_{r=1}^{N}\frac{1}{r^2}+\sum_{r=2}^{N+1}\frac{2}{r}+\sum_{r=2}^{N+1}\frac{1}{r^2}$$

M1

$$=-2+\frac{2}{N+1}+2\sum_{r=1}^{N}\frac{1}{r^2}-1+\frac{1}{(N+1)^2}$$

That is

$$\sum_{r=1}^{N}\frac{1}{r^2(r+1)^2}=2S_N-3+\frac{2}{N+1}+\frac{1}{(N+1)^2}$$

A1

From part (i),

$$\sum_{r=1}^{N}\frac{1}{r^2(r+1)}=\frac{1}{N+1}-1+\sum_{r=1}^{N}\frac{1}{r^2}=S_N-1+\frac{1}{N+1}$$

Thus

$$\sum_{r=1}^{N}\frac{1}{r^2(r+1)^2}=2\left(\sum_{r=1}^{N}\frac{1}{r^2(r+1)}+1-\frac{1}{N+1}\right)-3+\frac{2}{N+1}+\frac{1}{(N+1)^2}$$

M1

$$=2\sum_{r=1}^{N}\frac{1}{r^2(r+1)}-1+\frac{1}{(N+1)^2}$$

A1

Letting $N\to\infty$,

$$\sum_{r=1}^{\infty}\frac{1}{r^2(r+1)^2}=\sum_{r=1}^{\infty}\frac{2}{r^2(r+1)}-1$$

B1* (5)

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