STEP III, 2023, Q8 MS

8. (i)

$$y = xe^{-x}$$

$$\frac{dy}{dx} = e^{-x} - xe^{-x} = (1-x)e^{-x}$$

$$\frac{d^2y}{dx^2} = -e^{-x} - (1-x)e^{-x} = (x-2)e^{-x}$$

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = (x-2)e^{-x} + 2(1-x)e^{-x} + xe^{-x} = 0$$

M1 A1

$$x = 0, \qquad y = xe^{-x} = 0, \qquad \frac{dy}{dx} = (1-x)e^{-x} = 1$$

B1*

For $x \leq 1$, $(1-x) \geq 0$, $e^{-x} > 0$, so $\frac{dy}{dx} = (1-x)e^{-x} \geq 0$ **E1 (4)**

(ii) From (i),

$$g_1(x) = xe^{-x}$$

B1

Consider

$$y = g_2(x) = (a+bx)e^{x}$$

for $x \geq 1$ **B1**

Then g_2 must be a solution of $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$, $g_1(1) = g_2(1)$, and $g'_1(1) = g'_2(1)$

$$\frac{dy}{dx} = be^{x} + (a+bx)e^{x} = \big((a+b) + bx\big)e^{x}$$

$$\frac{d^2y}{dx^2} = be^{x} + \big((a+b) + bx\big)e^{x} = \big((a+2b) + bx\big)e^{x}$$

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = \big((a+2b) + bx\big)e^{x} - 2\big((a+b) + bx\big)e^{x} + (a+bx)e^{x} = 0$$

as required.

$$g_1(1) = g_2(1) \Rightarrow e^{-1} = (a+b)e$$

$$g'_1(1) = g'_2(1) \Rightarrow 0 = (a+2b)e$$

M1 A1

So $a = -2b$ and thus $b = -e^{-2}$

Hence, $g_2(x) = (2e^{-2} - e^{-2}x)e^{x} = (2-x)e^{x-2}$ **A1ft (5)**

(iii) $y = g_2(x)$ is a reflection of $y = g_1(x)$ in $x = 1$, **B1** which can be justified by substituting for x using $x' = 2 - x$ in $y = g_1(x)$, $xe^{-x} = (2-x')e^{x'-2}$ as expected. **E1 (2)**

(iv) If $y = k(c-x)$, then $\frac{dy}{dx} = -k'(c-x)$, and $\frac{d^2y}{dx^2} = k''(c-x)$ **M1**

So $\frac{d^2y}{dx^2} - p\frac{dy}{dx} + qy = \mathrm{k}''(c-x) + p\mathrm{k}'(c-x) + q\mathrm{k}(c-x) = 0$ **A1** provided that $r \le c - x \le s$

i.e. $c - s \le x \le c - r$ **B1 (3)**

(v) If $\mathrm{h}(x) = e^{-x}\sin x$, then $\mathrm{h}'(x) = -e^{-x}\sin x + e^{-x}\cos x$,

so $\mathrm{h}'\left(\frac{\pi}{4}\right) = -e^{-\frac{\pi}{4}}\sin\frac{\pi}{4} + e^{-\frac{\pi}{4}}\cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}}e^{-\frac{\pi}{4}} + \frac{1}{\sqrt{2}}e^{-\frac{\pi}{4}} = 0$ as required. **B1***

(a) Using (iv), the solution for $\frac{1}{4}\pi \le x \le \frac{5}{4}\pi$ must be $y = e^{-(c-x)}\sin(c-x)$ **M1** where

$-\frac{3}{4}\pi \le (c-x) \le \frac{1}{4}\pi$. **M1** That is $c = \frac{1}{2}\pi$. So $y = e^{x-\frac{1}{2}\pi}\cos x$ **A1**

(b) Similarly, the solution for $\frac{5}{4}\pi \le x \le \frac{9}{4}\pi$ must be $y = e^{c-x-\frac{1}{2}\pi}\cos(c-x)$ where

$\frac{1}{4}\pi \le (c-x) \le \frac{5}{4}\pi$. That is $c = \frac{5}{2}\pi$. **B1** So $y = e^{2\pi-x}\sin x$ **B1 (6)**