STEP III, 2023, Q8

8 If

$$y = \begin{cases} k_1(x) & x \leqslant b \\ k_2(x) & x \geqslant b \end{cases}$$

with $k_1(b) = k_2(b)$, then y is said to be *continuously differentiable* at $x = b$ if $k_1'(b) = k_2'(b)$.

(i) Let $f(x) = xe^{-x}$. Verify that, for all real x, $y = f(x)$ is a solution to the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$$

and that $y = 0$ and $\dfrac{dy}{dx} = 1$ when $x = 0$.

Show that $f'(x) \geqslant 0$ for $x \leqslant 1$.

(ii) You are given the differential equation

$$\frac{d^2y}{dx^2} + 2\left|\frac{dy}{dx}\right| + y = 0$$

where $y = 0$ and $\dfrac{dy}{dx} = 1$ when $x = 0$. Let

$$y = \begin{cases} g_1(x) & x \leqslant 1 \\ g_2(x) & x \geqslant 1 \end{cases}$$

be a solution of the differential equation which is continuously differentiable at $x = 1$.

Write down an expression for $g_1(x)$ and find an expression for $g_2(x)$.

(iii) State the geometrical relationship between the curves $y = g_1(x)$ and $y = g_2(x)$.

(iv) Prove that if $y = k(x)$ is a solution of the differential equation

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0$$

in the interval $r \leqslant x \leqslant s$, where p and q are constants, then, in a suitable interval which you should state, $y = k(c - x)$ satisfies the differential equation

$$\frac{d^2y}{dx^2} - p\frac{dy}{dx} + qy = 0\,.$$

THIS QUESTION CONTINUES ON THE FACING PAGE.

(v) You are given the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\left|\frac{\mathrm{d}y}{\mathrm{d}x}\right| + 2y = 0$$

where $y = 0$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 1$ when $x = 0$.

Let $\mathrm{h}(x) = \mathrm{e}^{-x}\sin x$. Show that $\mathrm{h}'\left(\frac{1}{4}\pi\right) = 0$.

It is given that $y = \mathrm{h}(x)$ satisfies the differential equation in the interval $-\frac{3}{4}\pi \leqslant x \leqslant \frac{1}{4}\pi$ and that $\mathrm{h}'(x) \geqslant 0$ in this interval.

In a solution to the differential equation which is continuously differentiable at $(n+\frac{1}{4})\pi$ for all $n \in \mathbb{Z}$, find y in terms of x in the intervals

(a) $\frac{1}{4}\pi \leqslant x \leqslant \frac{5}{4}\pi$,

(b) $\frac{5}{4}\pi \leqslant x \leqslant \frac{9}{4}\pi$.