

STEP III, 2023, Q7 MS

7. (i)

Let $x = u^2$, $\frac{dx}{du} = 2u$, $\sqrt{x} = u$ **M1**

$$\int_0^1 f(\sqrt{x}) dx = \int_0^1 f(u) 2u du = 2 \int_0^1 xf(x) dx$$

as required. **A1* (2)**

(ii)

$$\int_0^1 (g(x) - x)^2 dx = \int_0^1 (g(x))^2 dx - 2 \int_0^1 xg(x) dx + \int_0^1 x^2 dx$$

M1

$$= \int_0^1 g(\sqrt{x}) dx - \frac{1}{3} - 2 \int_0^1 xg(x) dx + \int_0^1 x^2 dx$$

$$= 2 \int_0^1 xg(x) dx - \frac{1}{3} - 2 \int_0^1 xg(x) dx + \left[\frac{x^3}{3} \right]_0^1$$

M1

$$= 0 - \frac{1}{3} + \frac{1}{3} = 0$$

A1*

$$(g(x) - x)^2 \geq 0$$

So, the area under the graph of $y = (g(x) - x)^2 \geq 0$, and the area can only equal zero if $(g(x) - x)^2 = 0$ for $0 \leq x \leq 1$, that is $g(x) = x$. **E1 (4)**

(iii)

$$\int_0^1 (h'(x) - x)^2 dx = \int_0^1 (h'(x))^2 - 2xh'(x) + x^2 dx$$

M1

We are given that

$$\int_0^1 (h'(x))^2 = 2h(1) - 2 \int_0^1 h(x) dx - \frac{1}{3}$$

Integrating by parts

$$\int_0^1 2xh'(x) dx = [2xh(x)]_0^1 - 2 \int_0^1 h(x) dx = 2h(1) - 2 \int_0^1 h(x) dx$$

M1 A1



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and

$$\int_0^1 x^2 dx = \left[\frac{x^3}{3} \right]_0^1 = \frac{1}{3}$$

So,

$$\int_0^1 (h'(x) - x)^2 dx = 2h(1) - 2 \int_0^1 h(x) dx - \frac{1}{3} - \left(2h(1) - 2 \int_0^1 h(x) dx \right) + \frac{1}{3} = 0$$

A1

As in (ii) with g , $h'(x) = x$. Thus $h(x) = \frac{1}{2}x^2 + c$ but $h(0) = 0$ so $c = 0$ and thus $h(x) = \frac{1}{2}x^2$

E1

M1 A1

A1 (8)

(iv)

$$\int_0^1 \left(e^{\frac{1}{2}ax} k(x) - e^{-\frac{1}{2}ax} \right)^2 dx = \int_0^1 e^{ax} (k(x))^2 - 2k(x) + e^{-ax} dx$$

M1

dM1

$$\begin{aligned} &= 2 \int_0^1 k(x) dx + \frac{e^{-a}}{a} - \frac{1}{a^2} - \frac{1}{4} - 2 \int_0^1 k(x) dx - \left[\frac{e^{-a}}{a} \right]_0^1 \\ &= \frac{e^{-a}}{a} - \frac{1}{a^2} - \frac{1}{4} - \frac{e^{-a}}{a} + \frac{1}{a} = -\frac{1}{a^2} + \frac{1}{a} - \frac{1}{4} = -\frac{4 - 4a + a^2}{4a^2} = -\frac{(2-a)^2}{4a^2} \end{aligned}$$

A1 **A1**

As before, $\int_0^1 \left(e^{\frac{1}{2}ax} k(x) - e^{-\frac{1}{2}ax} \right)^2 dx \geq 0$ but $-\frac{(2-a)^2}{4a^2} \leq 0$

Therefore, $\int_0^1 \left(e^{\frac{1}{2}ax} k(x) - e^{-\frac{1}{2}ax} \right)^2 dx = 0$ and $\frac{(2-a)^2}{4a^2} = 0$ **E1**

Thus $e^{\frac{1}{2}ax} k(x) - e^{-\frac{1}{2}ax} = 0$ and $2 - a = 0$

So $a = 2$ and $k(x) = e^{-ax} = e^{-2x}$ **A1 (6)**



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