STEP III, 2023, Q5 MS

5. (i)

$$\frac{1}{x}+\frac{2}{y}=\frac{2}{7}$$

$$7y+14x=2xy$$

$$2xy-7y-14x+49=49$$

$$(2x-7)(y-7)=49$$

B1*

Thus $2x-7=1$, $y-7=49$, or $2x-7=7$, $y-7=7$, or $2x-7=49$, $y-7=1$

M1

and so $(x,y)=(4,56)$, $(7,14)$, or $(28,8)$ **A1 (3)**

(ii)

$$p^2+pq+q^2=n^2$$

$$p^2+2pq+q^2-n^2=pq$$

$$(p+q)^2-n^2=pq$$

$$(p+q+n)(p+q-n)=pq$$

B1*

$p+q+n\neq p$ and $p+q+n\neq q$ as p, q, and n are all positive. $p+q+n>p+q-n$ so $p+q+n\neq 1$ as that would require $p+q-n=pq>1$. **M1**

Thus $p+q+n=pq$ and $p+q-n=1$ as required. **A1***

Therefore $p+q+p+q-1=pq$ **M1**

$$pq-2p-2q+4=3$$

$$(p-2)(q-2)=3$$

dM1

Thus $p-2=1$, $q-2=3$, or $p-2=3$, $q-2=1$

Alternative (I)

$$pq-2p-2q+4=3$$

$$pq-2p-2q+1=0$$

$$p(q-2)=2q-1$$

$$p=\frac{2q-1}{q-2}=2+\frac{3}{q-2}$$

as $q\neq 2$ ($q=2$ would yield -4+4-3=0) so $q-2=1\ or\ 3$ **E1**

and so $(p,q)=(3,5)$, or $(5,3)$ **A1 (6)**

Alternative (II) $p+q+n=pq$ and $p+q-n=1$ yield $p+q=n+1$ and $pq=2n+1$

Therefore, p and q are solutions of $t^2-(n+1)t+(2n+1)=0$

Hence $t=\frac{(n+1)\pm\sqrt{(n+1)^2-4(2n+1)}}{2}=\frac{(n+1)\pm\sqrt{(n-3)^2+12}}{2}$

For integer t we require that $(n-3)^2+12$ is a perfect square (in fact an even perfect square).

Thus the difference of squares between $(n-3)^2+12$ and $(n-3)^2$ is 12. Successive squares, z^2 and $(z+1)^2$ differ by $2z+1$, which for $z\geq 6$ is ≥ 13. Thus $(n-3)\leq 5$. Then, either by listing potential solutions exhaustively, or justifying that $(n-3)^2+12$ and $(n-3)^2$ have to be squares differing by $7+5$ and hence $(n-3)^2=2^2$ giving $(p,q)=(3,5)$, or $(5,3)$. **E1 A1 (6)**

(iii) If $p^3+q^3+3pq^2=n^3$, and as p, q, and hence n are all positive, then $p^3<n^3$ and $q^3<n^3$ so $p<n$ and $q<n$, **E1** and hence $p+q-n<p$ and $p+q-n<q$. **A1***

If

$$p^3+q^3+3pq^2+3p^2q=n^3+3p^2q$$

M1

$$(p+q)^3-n^3=3p^2q$$

dM1

$$(p+q-n)((p+q)^2+(p+q)n+n^2)=3p^2q$$

A1

As $p+q-n<p$ and $p+q-n<q$, so $p+q-n=1$ or 3 **A1**

If $p+q-n=1$ then $(n+1)^3-n^3=3p^2q$ and hence $3n^2+3n+1=3p^2q$ **M1** which is not possible as LHS is not a multiple of 3 and RHS is. **E1**

If $p+q-n=3$, then $(n+3)^3-n^3=3p^2q$ and hence $9n^2+27n+27=3p^2q$, that is $3(n^2+3n+3)=p^2q$. **M1** So p or q must divide 3 and hence must be 3 as p and q are prime. **E1**

If $p=3$, then $q-n=0$ but $q<n$ and vice versa if $q=3$ **E1* (11)**