STEP III, 2023, Q4 MS

4. (i) By de Moivre,

$$\cos\big((2n+1)\theta\big) + \mathrm{i}\sin\big((2n+1)\theta\big) = (\cos\theta + \mathrm{i}\sin\theta)^{2n+1}$$

Expanding by the binomial theorem and equating real parts

$$\cos\big((2n+1)\theta\big) = \cos^{2n+1}\theta - \binom{2n+1}{2}\cos^{2n-1}\theta\,\sin^2\theta + \cdots + (-1)^n\binom{2n+1}{2n}\cos\theta\ \sin^{2n}\theta$$

M1 A1

$$= \cos^{2n+1}\theta + \binom{2n+1}{2}\cos^{2n-1}\theta\,(\cos^2\theta - 1) + \cdots + \binom{2n+1}{2n}\cos\theta\ (\cos^2\theta - 1)^n$$

M1

$$= \sum_{r=0}^{n}\binom{2n+1}{2r}\cos^{2n+1-2r}\theta\,(\cos^2\theta - 1)^r$$

A1* (4)

Notice, for (iv), that this expression only contains odd powers of $\cos\theta$.

(ii) The coefficient of x^{2n+1} in $p(x)$ is

$$\sum_{r=0}^{n}\binom{2n+1}{2r}$$

B1

$$(1+x)^{2n+1} = \sum_{r=0}^{2n+1}\binom{2n+1}{r}x^r = \sum_{r=0}^{n}\binom{2n+1}{2r}x^{2r} + \sum_{r=0}^{n}\binom{2n+1}{2r+1}x^{2r+1}$$

Substituting $x = 1$,

$$2^{2n+1} = \sum_{r=0}^{n}\binom{2n+1}{2r} + \sum_{r=0}^{n}\binom{2n+1}{2r+1}$$

and substituting $x = -1$, **M1**

$$0 = \sum_{r=0}^{n}\binom{2n+1}{2r} - \sum_{r=0}^{n}\binom{2n+1}{2r+1}$$

A1

Adding these two results,

$$2^{2n+1} = 2\sum_{r=0}^{n}\binom{2n+1}{2r}$$

and so the required coefficient is 2^{2n} as required. **A1* (4)**

(iii) The coefficient of x^{2n-1} in $\mathrm{p}(x)$ is

$$\sum_{r=0}^{n} -r\binom{2n+1}{2r}$$

B1

As

$$(1+x)^{2n+1} = \sum_{r=0}^{2n+1}\binom{2n+1}{r}x^r = \sum_{r=0}^{n}\binom{2n+1}{2r}x^{2r} + \sum_{r=0}^{n}\binom{2n+1}{2r+1}x^{2r+1}$$

differentiating with respect to x

$$(2n+1)(1+x)^{2n} = \sum_{r=0}^{n} 2r\binom{2n+1}{2r}x^{2r-1} + \sum_{r=0}^{n}(2r+1)\binom{2n+1}{2r+1}x^{2r}$$

M1

Substituting $x = 1$,

$$(2n+1)2^{2n} = \sum_{r=0}^{n} 2r\binom{2n+1}{2r} + \sum_{r=0}^{n}(2r+1)\binom{2n+1}{2r+1}$$

M1

and substituting $x = -1$,

$$0 = -\sum_{r=0}^{n} 2r\binom{2n+1}{2r} + \sum_{r=0}^{n}(2r+1)\binom{2n+1}{2r+1}$$

A1

Subtracting these two results

$$(2n+1)2^{2n} = 2\sum_{r=0}^{n} 2r\binom{2n+1}{2r} = 4\sum_{r=0}^{n} r\binom{2n+1}{2r}$$

and so the required coefficient is

$$-(2n+1)2^{2n} \div 4 = -(2n+1)2^{2n-2}$$

A1* (5)

Alternative

$$\sum_{r=0}^{n} -r\binom{2n+1}{2r} = -\sum_{r=0}^{n} r\frac{(2n+1)!}{(2n-2r+1)!\,(2r)!} = -\frac{2n+1}{2}\sum_{r=1}^{n}\frac{(2n)!}{(2n-2r+1)!\,(2r-1)!}$$

M1

$$= -\frac{2n+1}{2}\sum_{r=1}^{n}\binom{2n}{2r-1} = -\frac{2n+1}{2}\sum_{r=0}^{n-1}\binom{2n}{2r+1}$$

M1

As in (ii),

$$\sum_{r=0}^{n-1}\binom{2n}{2r+1} = \frac{1}{2}2^{2n}$$

A1

so

$$\sum_{r=0}^{n} -r\binom{2n+1}{2r} = -\frac{2n+1}{2}\frac{1}{2}2^{2n} = -(2n+1)2^{2n-2}$$

A1* (5)

(iv) Suppose

$$q(x) = ax^n + bx^{n-1} + cx^{n-2} + \cdots$$

then

$$p(x) = (x+1)[ax^n + bx^{n-1} + cx^{n-2} + \cdots]^2$$
$$= (x+1)(a^2x^{2n} + 2abx^{2n-1} + (b^2 + 2ac)x^{2n-2} + \cdots)$$
$$= a^2x^{2n+1} + (a^2 + 2ab)x^{2n} + (b^2 + 2ac + 2ab)x^{2n-1} + \cdots$$

M1 A1

Thus $a^2 = 2^{2n}$, $a^2 + 2ab = 0$, and $b^2 + 2ac + 2ab = -(2n+1)2^{2n-2}$ **dM1 A1**

Therefore $a = 2^n$ (as $a > 0$), **B1**

$$b = \frac{-a}{2} = -2^{n-1}$$

A1

and

$$2^{2n-2} + 2^{n+1}c - 2^{2n} = -(2n+1)2^{2n-2}$$

so

$$2^{n-3} + c - 2^{n-1} = -(2n+1)2^{n-3}$$
$$c = 2^{n-3}(4 - 1 - 2n - 1) = 2^{n-2}(1-n)$$

as required. **A1*(7)**