STEP III, 2023, Q4

4 Let n be a positive integer. The polynomial p is defined by the identity

$$\mathrm{p}(\cos\theta) \equiv \cos\big((2n+1)\theta\big) + 1\,.$$

(i) Show that

$$\cos\big((2n+1)\theta\big) = \sum_{r=0}^{n} \binom{2n+1}{2r} \cos^{2n+1-2r}\theta(\cos^2\theta - 1)^r\,.$$

(ii) By considering the expansion of $(1+t)^{2n+1}$ for suitable values of t, show that the coefficient of x^{2n+1} in the polynomial $\mathrm{p}(x)$ is 2^{2n}.

(iii) Show that the coefficient of x^{2n-1} in the polynomial $\mathrm{p}(x)$ is $-(2n+1)2^{2n-2}$.

(iv) It is given that there exists a polynomial q such that

$$\mathrm{p}(x) = (x+1)\,[\mathrm{q}(x)]^2$$

and the coefficient of x^n in $\mathrm{q}(x)$ is greater than 0.

Write down the coefficient of x^n in the polynomial $\mathrm{q}(x)$ and, for $n \geqslant 2$, show that the coefficient of x^{n-2} in the polynomial $\mathrm{q}(x)$ is

$$2^{n-2}(1-n)\,.$$