STEP III, 2022, Q6 MS

6. (i)

$$\cos(\theta+\alpha)-\cos\theta=\cos\theta\cos\alpha-\sin\theta\sin\alpha-\cos\theta \approx \cos\theta\left(1-\frac{\alpha^2}{2}\right)-\sin\theta\,\alpha-\cos\theta$$

$=-\alpha\sin\theta-\frac{\alpha^2}{2}\cos\theta$ as required. **M1 *A1**

If $\sin\theta\neq 0$

$$\lim_{\alpha\to 0}\frac{\sin(\theta+\alpha)-\sin\theta}{\cos(\theta+\alpha)-\cos\theta}=\lim_{\alpha\to 0}\frac{\alpha\cos\theta-\frac{\alpha^2}{2}\sin\theta}{-\alpha\sin\theta-\frac{\alpha^2}{2}\cos\theta}=\lim_{\alpha\to 0}\frac{\cos\theta-\frac{\alpha}{2}\sin\theta}{-\sin\theta-\frac{\alpha}{2}\cos\theta}=-\cot\theta$$

M1 A1 **A1**

(Alternative by l'Hopital, $\lim_{\alpha\to 0}\frac{\sin(\theta+\alpha)-\sin\theta}{\cos(\theta+\alpha)-\cos\theta}=\lim_{\alpha\to 0}\frac{\cos(\theta+\alpha)}{-\sin(\theta+\alpha)}=\lim_{\alpha\to 0}-\cot(\theta+\alpha)=-\cot\theta$

M1 A1 **A1**)

If $\sin\theta=0$

$$\lim_{\alpha\to 0}\frac{\sin(\theta+\alpha)-\sin\theta}{\cos(\theta+\alpha)-\cos\theta}=\lim_{\alpha\to 0}\frac{\cos\theta}{-\frac{\alpha}{2}\cos\theta}=\lim_{\alpha\to 0}\frac{-2}{\alpha}$$

M1

$\to-\infty$ as $\alpha\to+0$ and $\to\infty$ as $\alpha\to-0$ **A1 (7)**

(ii) (a) If Q_0 is the initial point of contact of C_1 and C_2, and if X is the point on C_2 which was initially at Q_0, then if $QOQ_0=\theta$, arc QQ_0 on C_1 is of length $(n-1)a\theta$ **E1** and this will equal the arc length QX on C_2. So if T is the centre of C_2, $QTX=(n-1)\theta$, and TP makes an angle $\theta+(n-1)\theta=n\theta$ with the x axis. **E1**

Thus the x-coordinate of P is $x(\theta)=na\cos\theta+a\cos(n\theta)=a(n\cos\theta+\cos n\theta)$ as required.

Similarly, $y(\theta)=a(n\sin\theta+\sin n\theta)$. **M1 *A1 (4)**

(b) $OP=(n-1)a$ if and only if $(n\cos\theta+\cos n\theta)^2+(n\sin\theta+\sin n\theta)^2=(n-1)^2$

That is if $n^2+2n\cos(n-1)\theta+1=n^2-2n+1$ which is $\cos(n-1)\theta=-1$

M1

so, when $(n-1)\theta$ is an odd multiple of π **M1**

Therefore $\theta=\frac{2r+1}{n-1}\pi$ for $r=0,1,\cdots$ **A1 (3)**

(Alternatively, $OP=(n-1)a$ only if $n\cos\theta+\cos n\theta=(n-1)\cos\theta$ i.e. $\cos n\theta=-\cos\theta$, and $n\sin\theta+\sin n\theta=(n-1)\sin\theta$ i.e. $\sin n\theta=-\sin\theta$ **M1**

Thus $\cos(n-1)\theta=-\cos\theta\cos\theta+-\sin\theta\sin\theta=-1$ so $(n-1)\theta$ is an odd multiple of π **M1**

Result as before **A1**)

(c)

$$\lim_{\alpha\to 0}\frac{y(\theta_0+\alpha)-y(\theta_0)}{x(\theta_0+\alpha)-x(\theta_0)}=\lim_{\alpha\to 0}\frac{a(n\sin(\theta_0+\alpha)+\sin n(\theta_0+\alpha))-a(n\sin\theta_0+\sin n\theta_0)}{a(n\cos(\theta_0+\alpha)+\cos n(\theta_0+\alpha))-a(n\cos\theta_0+\cos n\theta_0)}$$

M1

$$=\lim_{\alpha\to 0}\frac{n\left(\alpha\cos\theta_0-\frac{\alpha^2}{2}\sin\theta_0\right)+\left(n\alpha\cos n\theta_0-\frac{n^2\alpha^2}{2}\sin n\theta_0\right)}{n\left(-\alpha\sin\theta_0-\frac{\alpha^2}{2}\cos\theta_0\right)+\left(-n\alpha\sin n\theta_0-\frac{n^2\alpha^2}{2}\cos n\theta_0\right)}$$

M1 A1

$$=\lim_{\alpha\to 0}\frac{\cos\theta_0+\cos n\theta_0-\frac{\alpha}{2}(\sin\theta_0+n\sin n\theta_0)}{-(\sin\theta_0+\sin n\theta_0)-\frac{\alpha}{2}(\cos\theta_0+n\cos n\theta_0)}$$

$$=\frac{\sin\theta_0+n\sin n\theta_0}{\cos\theta_0+n\cos n\theta_0}$$

as $\cos\theta_0+\cos n\theta_0=2\cos(n+1)\frac{\theta_0}{2}\cos(n-1)\frac{\theta_0}{2}$ and $(n-1)\frac{\theta_0}{2}=\frac{\pi}{2}$ so $\cos(n-1)\frac{\theta_0}{2}=0$

and similarly, $\sin\theta_0+\sin n\theta_0=2\sin(n+1)\frac{\theta_0}{2}\cos(n-1)\frac{\theta_0}{2}=0$

Further,

$$\sin\theta_0+n\sin n\theta_0=\sin\theta_0+n\big(\sin\big((n-1)+1\big)\theta_0\big)$$
$$=\sin\theta_0+n(\sin(n-1)\theta_0\cos\theta_0+\cos(n-1)\theta_0\sin\theta_0)$$
$$=(1-n)\sin\theta_0$$

and

$$\cos\theta_0+n\cos n\theta_0=\cos\theta_0+n(\cos(n-1)\theta_0\cos\theta_0-\sin(n-1)\theta_0\sin\theta_0)$$
$$=(1-n)\cos\theta_0$$

So

$$\lim_{\alpha\to 0}\frac{y(\theta_0+\alpha)-y(\theta_0)}{x(\theta_0+\alpha)-x(\theta_0)}=\frac{(1-n)\sin\theta_0}{(1-n)\cos\theta_0}=\tan\theta_0$$

M1 **A1**

The LHS is the gradient of the tangent to the curve at P and the RHS is the gradient of OP, as required. **E1 (6)**