STEP III, 2022, Q5 MS

5. (i)

$$\int_{-a}^{a} \frac{1}{1+e^x}\,dx = \int_{-a}^{a} \frac{e^{-x}}{e^{-x}+1}\,dx = [-\ln(e^{-x}+1)]_{-a}^{a} = \ln\left(\frac{e^a+1}{e^{-a}+1}\right) = \ln e^a = a$$

M1 **A1** ***A1(3)**

Alternative 1.

$\int_{-a}^{a} \frac{1}{1+e^x}\,dx = \int_{-a}^{a} \frac{1+e^x}{1+e^x} - \frac{e^x}{1+e^x}\,dx = [x - \ln(1+)]_{-a}^{a} = 2a - \ln\left(\frac{e^a+1}{e^{-a}+1}\right) = 2a - \ln e^a = a$

M1 **A1** ***A1(3)**

Alternative 2.

Substitute $u = e^x$,

$\int_{-a}^{a} \frac{1}{1+e^x}\,dx = \int_{e^{-a}}^{e^a} \frac{1}{1+u}\frac{1}{u}\,du = \int_{e^{-a}}^{e^a} \frac{1}{u} - \frac{1}{1+u}\,du = [\ln u - \ln(1+u)]_{e^{-a}}^{e^a} = 2a - \ln\left(\frac{e^a+1}{e^{-a}+1}\right) =$

M1 **A1**

$2a - \ln e^a = a$

***A1(3)**

Alternative 3.

Substitute $u = 1 + e^x$,

$\int_{-a}^{a} \frac{1}{1+e^x}\,dx = \int_{1+e^{-a}}^{1+e^a} \frac{1}{u}\frac{1}{u}\,du = \int_{e^{-a}}^{e^a} \frac{1}{u} - \frac{1}{1+u}\,du = [\ln u - \ln(1+u)]_{e^{-a}}^{e^a} = 2a - \ln\left(\frac{e^a+1}{e^{-a}+1}\right) =$

M1 **A1**

$2a - \ln e^a = a$

***A1(3)**

Alternative 4.

$$\int_{-a}^{a} \frac{1}{1+e^x}\,dx = \int_{0}^{a} \frac{1}{1+e^x}\,dx + \int_{-a}^{0} \frac{1}{1+e^x}\,dx = \int_{0}^{a} \frac{1}{1+e^x}\,dx + \int_{a}^{0} \frac{1}{1+e^{-x}}.-dx$$

$$= \int_{0}^{a} \frac{1}{1+e^x}\,dx + \int_{0}^{a} \frac{1}{1+e^{-x}}\,dx = \int_{0}^{a} \frac{1}{1+e^x} + \frac{1}{1+e^{-x}}\,dx = \int_{0}^{a} \frac{1+e^{-x}+1+e^{-x}}{(1+e^x)(1+e^{-x})}\,dx$$

M1

$$= \int_{0}^{a} \frac{2+e^{-x}+e^{-x}}{2+e^{-x}+e^{-x}}\,dx = [x]_0^a = a$$

A1 ***A1 (3)**

(ii)

Suppose

$$\int g(x)\ dx\ = G(x) + c$$

Then if

$$\int_0^a g(x)\ dx = 0 \quad \forall a \ge 0$$

$$G(a) - G(0) = 0 \quad \forall a$$

so $G(a) = constant\ \ \forall a$ and hence $\frac{dG}{dx} = g(x) = 0\ \ \forall x \ge 0$ as required.

Alternatively, by the FTC, $g(a) = 0\ \ \forall a \ge 0$ **E1**

$$\int_{-a}^{a} \frac{1}{1+f(x)}\ dx = a \ \Leftrightarrow\ \int_{-a}^{0} \frac{1}{1+f(x)}\ dx + \int_0^a \frac{1}{1+f(x)}\ dx = a$$

M1

$$\Leftrightarrow \int_a^0 \frac{1}{1+f(-x)} \cdot - dx + \int_0^a \frac{1}{1+f(x)}\ dx = a$$

M1 A1

$$\Leftrightarrow \int_0^a \frac{1}{1+f(-x)} + \frac{1}{1+f(x)} - 1\ \ dx = 0$$

M1 A1

so, by stated result at start of part,

$$\Leftrightarrow \frac{1}{1+f(-x)} + \frac{1}{1+f(x)} - 1\ = 0 \quad \forall x$$

E1 E1

$$\Leftrightarrow 1 + f(x) + 1 + f(-x) - \big(1 + f(-x)\big)\big(1 + f(x)\big) = 0$$

$$\Leftrightarrow f(x)\,f(-x) = 1$$

***B1 (9)**

(iii)

$$\int_{-a}^{a} \frac{h(x)}{1+f(x)}\ dx = \int_{-a}^{0} \frac{h(x)}{1+f(x)}\ dx + \int_0^a \frac{h(x)}{1+f(x)}\ dx = \int_a^0 \frac{h(-x)}{1+f(-x)} \cdot -dx + \int_0^a \frac{h(x)}{1+f(x)}\ dx$$

M1

$$= \int_0^a \frac{h(x)}{1+f(-x)} + \frac{h(x)}{1+f(x)}\, dx = \int_0^a h(x)\, dx$$

by the result of (ii). **M1 *A1 (3)**

(iv)

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{e^{-x}\cos x}{\cosh x}\, dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{e^{-x}\cos x}{\frac{e^x+e^{-x}}{2}}\, dx = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos x}{1+e^{2x}}\, dx$$

M1 A1

$\cos x$ satisfies the conditions for $h(x)$ in part (iii) and e^{2x} satisfies the conditions for $f(x)$ in part (ii). **E1**

Therefore,

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{e^{-x}\cos x}{\cosh x}\, dx = 2\int_0^{\frac{\pi}{2}} \cos x\, dx = 2\,[\sin x]_0^{\frac{\pi}{2}} = 2$$

M1 **A1 (5)**