STEP III, 2022, Q5

5 **(i)** Show that

$$\int_{-a}^{a} \frac{1}{1+e^x}\,dx = a \quad \text{for all } a \geqslant 0.$$

(ii) Explain why, if g is a continuous function and

$$\int_{0}^{a} g(x)\,dx = 0 \quad \text{for all } a \geqslant 0,$$

then $g(x) = 0$ for all $x \geqslant 0$.

Let f be a continuous function with $f(x) \geqslant 0$ for all x. Show that

$$\int_{-a}^{a} \frac{1}{1+f(x)}\,dx = a \quad \text{for all } a \geqslant 0$$

if and only if

$$\frac{1}{1+f(x)} + \frac{1}{1+f(-x)} - 1 = 0 \quad \text{for all } x \geqslant 0,$$

and hence if and only if $f(x)f(-x) = 1$ for all x.

(iii) Let f be a continuous function such that, for all x, $f(x) \geqslant 0$ and $f(x)f(-x) = 1$. Show that, if h is a continuous function with $h(x) = h(-x)$ for all x, then

$$\int_{-a}^{a} \frac{h(x)}{1+f(x)}\,dx = \int_{0}^{a} h(x)\,dx\,.$$

(iv) Hence find the exact value of

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{e^{-x}\cos x}{\cosh x}\,dx\,.$$