STEP III, 2022, Q4 MS

4. (i) Suppose

$$2^k \cosh\frac{x}{2}\cosh\frac{x}{4}\cdots\cosh\frac{x}{2^k}\sinh\frac{x}{2^k} = \sinh x$$

for some integer k . **E1**

Then

$$2^{k+1}\cosh\frac{x}{2}\cosh\frac{x}{4}\cdots\cosh\frac{x}{2^{k+1}}\sinh\frac{x}{2^{k+1}} = 2\sinh x\frac{\cosh\frac{x}{2^{k+1}}\sinh\frac{x}{2^{k+1}}}{\sinh\frac{x}{2^k}}$$

(which is legitimate because $x \neq 0$ and hence $\sinh\frac{x}{2^k} \neq 0$)

$$= \sinh x\,\frac{2\sinh\frac{x}{2^{k+1}}\cosh\frac{x}{2^{k+1}}}{\sinh\frac{x}{2^k}} = \sinh x\,\frac{\sinh\frac{x}{2^k}}{\sinh\frac{x}{2^k}} = \sinh x$$

which is the desired result for $k+1$. **M1**

$$2\cosh\frac{x}{2}\sinh\frac{x}{2} = \sinh x$$

B1

so, the result is true for $n = 1$.

Hence by the principle of mathematical induction,

$$\sinh x = 2^n\cosh\frac{x}{2}\cosh\frac{x}{4}\cdots\cosh\frac{x}{2^n}\sinh\frac{x}{2^n}$$

for all positive integer n .

Thus

$$\frac{\sinh x}{x}\,\frac{\frac{x}{2^n}}{\sinh\frac{x}{2^n}} = 2^n\cosh\frac{x}{2}\cosh\frac{x}{4}\cdots\cosh\frac{x}{2^n}\sinh\frac{x}{2^n}\,\frac{x}{x}\,\frac{1}{2^n}\,\frac{1}{\sinh\frac{x}{2^n}} = \cosh\frac{x}{2}\cosh\frac{x}{4}\cdots\cosh\frac{x}{2^n}$$

as required. This working is permissible as $x \neq 0$, and so $\sinh\frac{x}{2^k} \neq 0$. **E1(4)**

(ii)

$$\frac{y}{\sinh y} = \frac{y}{y + \frac{y^3}{3!} + \frac{y^5}{5!} + \cdots} = \frac{1}{1 + \frac{y^2}{3!} + \frac{y^4}{5!} + \cdots} \to 1$$

as $y \to 0$. **E1**

As, from (i),

$$\frac{\sinh x}{x}\,\frac{\frac{x}{2^n}}{\sinh\frac{x}{2^n}} = \cosh\frac{x}{2}\cosh\frac{x}{4}\cdots\cosh\frac{x}{2^n}$$

letting $n \to \infty$,

and using the result shown from the use of the Maclaurin series that

$$\frac{\frac{x}{2^n}}{\sinh\frac{x}{2^n}} \to 1$$

we have

$$\frac{\sinh x}{x} = \cosh\frac{x}{2}\cosh\frac{x}{4}\cdots\cosh\frac{x}{2^n}\cdots$$

as required. **E1 (2)**

(iii) Letting $x = \ln 2$, $\sinh x = \frac{2-\frac{1}{2}}{2} = \frac{3}{4}$, $\cosh\frac{x}{2} = \frac{\sqrt{2}+\frac{1}{\sqrt{2}}}{2} = \frac{3}{2\sqrt{2}}$, $\cosh\frac{x}{4} = \frac{\sqrt{\sqrt{2}}+\frac{1}{\sqrt{\sqrt{2}}}}{2} = \frac{\sqrt{2}+1}{2\sqrt{\sqrt{2}}}$

M1

etc.

Thus

$$\frac{\frac{3}{4}}{\ln 2} = \frac{3}{2\sqrt{2}} \times \frac{\sqrt{2}+1}{2\sqrt{\sqrt{2}}} \times \frac{\sqrt{\sqrt{2}}+1}{2\sqrt{\sqrt{\sqrt{2}}}}\cdots$$

M1 A1

and

$$\frac{1}{\ln 2} = \frac{4}{2\sqrt{2}} \times \frac{\sqrt{2}+1}{2\sqrt{\sqrt{2}}} \times \frac{\sqrt{\sqrt{2}}+1}{2\sqrt{\sqrt{\sqrt{2}}}}\cdots = \frac{4}{2\sqrt{2}\sqrt{\sqrt{2}}\sqrt{\sqrt{\sqrt{2}}}\cdots} \times \frac{1+\sqrt{2}}{2} \times \frac{1+\sqrt{\sqrt{2}}}{2} \times\cdots$$

A1

The denominator of the first fraction is

$$2 \times 2^{\frac{1}{2}} \times 2^{\frac{1}{4}} \times\cdots = 2^{1+\frac{1}{2}+\frac{1}{4}+\cdots} = 2^2 = 4$$

E1

So

$$\frac{1}{\ln 2} = \frac{1+\sqrt{2}}{2} \times \frac{1+\sqrt{\sqrt{2}}}{2} \times\cdots$$

as required.

(iv) Substituting $x = \frac{i\pi}{2}$ in **M1**

$$\frac{\sinh x}{x} = \cosh\frac{x}{2}\cosh\frac{x}{4}\cdots\cosh\frac{x}{2^n}\cdots$$

and using $\sinh ix = i\sin x$, $\cosh ix = \cos x$, **M1**

$$\frac{\sinh\frac{i\pi}{2}}{\frac{i\pi}{2}} = \cosh\frac{i\pi}{4}\cosh\frac{i\pi}{8}\cdots\cosh\frac{i\pi}{2^{n+1}} = \frac{i}{\frac{i\pi}{2}} = \cos\frac{\pi}{4}\cos\frac{\pi}{8}\cdots\cos\frac{\pi}{2^{n+1}}\cdots$$

M1 A1 A1

$\cos\frac{\pi}{4}=\frac{\sqrt{2}}{2}$, $\cos\frac{\pi}{4}=2\cos^2\frac{\pi}{8}-1$ and thus $\cos\frac{\pi}{8}=\sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}}=\frac{\sqrt{2+\sqrt{2}}}{2}$

and similarly, $\cos\frac{\pi}{16}=\sqrt{\frac{1+\frac{\sqrt{2+\sqrt{2}}}{2}}{2}}=\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}$ etc. M1 A1 M1

So

$$\frac{2}{\pi}=\frac{\sqrt{2}}{2}\times\frac{\sqrt{2+\sqrt{2}}}{2}\times\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\cdots$$

as required. *A1 (9)

(Alternatively, by induction

$$\sin x=2^n\cos\frac{x}{2}\cos\frac{x}{4}\cdots\cos\frac{x}{2^n}\sin\frac{x}{2^n}$$

M1 A1 E1 (as for (i)

As $\frac{y}{\sin y}\to 1$ as $y\to 0$,

$$\frac{\sin x}{x}=\cos\frac{x}{2}\cos\frac{x}{4}\cdots\cos\frac{x}{2^n}\cdots$$

M1A1

and then, substituting $x=\frac{\pi}{2}$ M1 result follows as before A1M1A1.)