

## STEP III, 2022, Q4

4 You may assume that all infinite sums and products in this question converge.

(i) Prove by induction that for all positive integers  $n$ ,

$$\sinh x = 2^n \cosh\left(\frac{x}{2}\right) \cosh\left(\frac{x}{4}\right) \cdots \cosh\left(\frac{x}{2^n}\right) \sinh\left(\frac{x}{2^n}\right)$$

and deduce that, for  $x \neq 0$ ,

$$\frac{\sinh x}{x} \frac{\frac{x}{2^n}}{\sinh\left(\frac{x}{2^n}\right)} = \cosh\left(\frac{x}{2}\right) \cosh\left(\frac{x}{4}\right) \cdots \cosh\left(\frac{x}{2^n}\right).$$

(ii) You are given that the Maclaurin series for  $\sinh x$  is

$$\sinh x = \sum_{r=0}^{\infty} \frac{x^{2r+1}}{(2r+1)!}.$$

Use this result to show that, as  $y$  tends to 0,  $\frac{y}{\sinh y}$  tends to 1.

Deduce that, for  $x \neq 0$ ,

$$\frac{\sinh x}{x} = \cosh\left(\frac{x}{2}\right) \cosh\left(\frac{x}{4}\right) \cdots \cosh\left(\frac{x}{2^n}\right) \cdots.$$

(iii) Let  $x = \ln 2$ . Evaluate  $\cosh\left(\frac{x}{2}\right)$  and show that

$$\cosh\left(\frac{x}{4}\right) = \frac{1 + 2^{\frac{1}{2}}}{2 \times 2^{\frac{1}{4}}}.$$

Use part (ii) to show that

$$\frac{1}{\ln 2} = \frac{1 + 2^{\frac{1}{2}}}{2} \times \frac{1 + 2^{\frac{1}{4}}}{2} \times \frac{1 + 2^{\frac{1}{8}}}{2} \cdots.$$

(iv) Show that

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \times \frac{\sqrt{2 + \sqrt{2}}}{2} \times \frac{\sqrt{2 + \sqrt{2 + \sqrt{2}}}}{2} \cdots.$$



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