STEP III, 2022, Q4

4 *You may assume that all infinite sums and products in this question converge.*

(i) Prove by induction that for all positive integers n,

$$\sinh x = 2^n \cosh\left(\frac{x}{2}\right)\cosh\left(\frac{x}{4}\right)\cdots\cosh\left(\frac{x}{2^n}\right)\sinh\left(\frac{x}{2^n}\right)$$

and deduce that, for $x \neq 0$,

$$\frac{\sinh x}{x}\,\frac{\frac{x}{2^n}}{\sinh\left(\frac{x}{2^n}\right)} = \cosh\left(\frac{x}{2}\right)\cosh\left(\frac{x}{4}\right)\cdots\cosh\left(\frac{x}{2^n}\right).$$

(ii) You are given that the Maclaurin series for $\sinh x$ is

$$\sinh x = \sum_{r=0}^{\infty} \frac{x^{2r+1}}{(2r+1)!}.$$

Use this result to show that, as y tends to 0, $\dfrac{y}{\sinh y}$ tends to 1.

Deduce that, for $x \neq 0$,

$$\frac{\sinh x}{x} = \cosh\left(\frac{x}{2}\right)\cosh\left(\frac{x}{4}\right)\cdots\cosh\left(\frac{x}{2^n}\right)\cdots.$$

(iii) Let $x = \ln 2$. Evaluate $\cosh\left(\frac{x}{2}\right)$ and show that

$$\cosh\left(\frac{x}{4}\right) = \frac{1+2^{\frac{1}{2}}}{2\times 2^{\frac{1}{4}}}.$$

Use part **(ii)** to show that

$$\frac{1}{\ln 2} = \frac{1+2^{\frac{1}{2}}}{2}\times\frac{1+2^{\frac{1}{4}}}{2}\times\frac{1+2^{\frac{1}{8}}}{2}\cdots.$$

(iv) Show that

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2}\times\frac{\sqrt{2+\sqrt{2}}}{2}\times\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\cdots.$$