STEP III, 2022, Q2

2 **(i)** Suppose that there are three non-zero integers a, b and c for which $a^3 + 2b^3 + 4c^3 = 0$. Explain why there must exist an integer p, with $|p| < |a|$, such that $4p^3 + b^3 + 2c^3 = 0$, and show further that there must exist integers p, q and r, with $|p| < |a|$, $|q| < |b|$ and $|r| < |c|$, such that $p^3 + 2q^3 + 4r^3 = 0$. Deduce that no such integers a, b and c can exist.

(ii) Prove that there are no non-zero integers a, b and c for which $9a^3 + 10b^3 + 6c^3 = 0$.

(iii) By considering the expression $(3n \pm 1)^2$, prove that, unless an integer is a multiple of three, its square is one more than a multiple of 3. Deduce that the sum of the squares of two integers can only be a multiple of three if each of the integers is a multiple of three.

Hence prove that there are no non-zero integers a, b and c for which $a^2 + b^2 = 3c^2$.

(iv) Prove that there are no non-zero integers a, b and c for which $a^2 + b^2 + c^2 = 4abc$.