STEP III, 2022, Q1

1 Let C_1 be the curve given by the parametric equations

$$x = ct\,, \quad y = \frac{c}{t}\,,$$

where $c > 0$ and $t \neq 0$, and let C_2 be the circle

$$(x-a)^2 + (y-b)^2 = r^2\,.$$

C_1 and C_2 intersect at the four points P_i $(i = 1, 2, 3, 4)$, and the corresponding values of the parameter t at these points are t_i.

(i) Show that t_i are the roots of the equation

$$c^2t^4 - 2act^3 + (a^2 + b^2 - r^2)t^2 - 2bct + c^2 = 0\,. \qquad (*)$$

(ii) Show that

$$\sum_{i=1}^{4} t_i^2 = \frac{2}{c^2}(a^2 - b^2 + r^2)$$

and find a similar expression for $\displaystyle\sum_{i=1}^{4} \frac{1}{t_i^2}$.

(iii) Hence show that $\displaystyle\sum_{i=1}^{4} OP_i^2 = 4r^2$, where OP_i denotes the distance of the point P_i from the origin.

(iv) Suppose that the curves C_1 and C_2 touch at two distinct points.

By considering the product of the roots of $(*)$, or otherwise, show that the centre of circle C_2 must lie on either the line $y = x$ or $y = -x$.