STEP III, 2021, Q1 MS

1. (i) $x = -4\cos^3 t$ so $\frac{dx}{dt} = 12\cos^2 t\ \sin t$ **M1**

$y = 12\sin t - 4\ \sin^3 t$ so $\frac{dy}{dt} = 12\cos t - 12\sin^2 t\ \cos t = 12\cos t\ (1 - \sin^2 t) = 12\cos^3 t$

M1

So $\frac{dy}{dx} = \frac{12\cos^3 t}{12\cos^2 t\ \sin t} = \cot t$ **A1**

Thus the equation of the normal at $(-4\cos^3\varphi\ , 12\sin\varphi - 4\ \sin^3\varphi\ )$ is

$$y - (12\sin\varphi - 4\ \sin^3\varphi) = -\frac{1}{\cot\varphi}(x - -4\cos^3\varphi)$$

M1 A1ft

This simplifies to $x\sin\varphi + y\cos\varphi = 12\sin\varphi\cos\varphi - 4\sin^3\varphi\cos\varphi - 4\sin\varphi\cos^3\varphi$

That is $x\sin\varphi + y\cos\varphi = 8\sin\varphi\cos\varphi$ **A1 (6)**

Alternative simplification $x\tan\varphi + y = 8\sin\varphi$

For $x = 8\cos^3 t$, $\frac{dx}{dt} = -24\cos^2 t\ \sin t$ and for $y = 8\ \sin^3 t$, $\frac{dy}{dt} = 24\sin^2 t\ \cos t$

So $\frac{dy}{dx} = \frac{24\sin^2 t\ \cos t}{-24\cos^2 t\ \sin t} = -\tan t$ **M1 A1ft**

Thus the equation of the tangent to $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 4$ at $(8\cos^3\varphi\ , 8\ \sin^3\varphi\ )$ is

$$y - 8\ \sin^3\varphi = -\tan\varphi\ (x - 8\cos^3\varphi)$$

M1

This simplifies to

$$x\sin\varphi + y\cos\varphi = 8\ \sin^3\varphi\cos\varphi + \ 8\sin\varphi\cos^3\varphi = 8\sin\varphi\cos\varphi\ (\sin^2\varphi + \cos^2\varphi)$$

That is $x\sin\varphi + y\cos\varphi = 8\sin\varphi\cos\varphi$ as required. **A1 (4)**

Alternative 1

the normal is a tangent to the second curve if it has the same gradient and the point $(8\cos^3\varphi\ , 8\ \sin^3\varphi\ )$ lies on the normal. **M1**

Gradient working as before **M1A1ft**

Substitution $x\sin\varphi + y\cos\varphi = 8\sin\varphi\ \cos^3\varphi + 8\ \sin^3\varphi\ \cos\varphi = 8\sin\varphi\cos\varphi\,(\sin^2\varphi + \cos^2\varphi) = 8\sin\varphi\cos\varphi$ as required or $x\tan\varphi + y = 8\sin\varphi\cos\varphi\,(\sin^2\varphi + \cos^2\varphi)$ **A1**

Alternative 2

$$\frac{2}{3}x^{\frac{-1}{3}} + \frac{2}{3}y^{\frac{-1}{3}}\frac{dy}{dx} = 0$$

M1

(ii) $x = \cos t + t\sin t$ so $\frac{dx}{dt} = -\sin t + t\cos t + \sin t = t\cos t$

$y = \sin t - t\ \cos t$ so $\frac{dy}{dt} = \ \cos t - \cos t + t\sin t = t\sin t$ **M1**

So $\frac{dy}{dx} = \tan t$ **A1**

Thus the equation of the normal at $(\cos\varphi + \ \varphi\sin\ \varphi\ ,\ \sin\ \varphi - \ \varphi\ \cos\ \varphi)$ is

$$y - (\sin\ \varphi - \ \varphi\ \cos\ \varphi) = -\cot\varphi(x - (\cos\varphi + \ \varphi\sin\ \varphi))$$

M1 A1ft

This simplifies to $x\cos\varphi + y\ \sin\ \varphi = 1$ **A1 (5)**

Alternatives which can be followed through to perpendicular distance step, or alternative method # are

$x + y\ \tan\varphi = \sec\varphi$ and $x\cot\varphi + y\ = \csc\varphi$

The distance of $(0,0)$ from $x\cos\varphi + y\ \sin\ \varphi = 1$ is $\left|\frac{-1}{\sqrt{\cos^2\varphi+\sin^2\varphi}}\right| = 1$

M1 A1ft **A1**

Alternatively, the perpendicular to $x\cos\varphi + y\ \sin\ \varphi = 1$ through $(0,0)$ is

$y\cos\varphi - x\ \sin\ \varphi = 0$, and these two lines meet at $(\cos\varphi\ ,\ \sin\ \varphi)$

M1 A1ft

which is a distance $\sqrt{\cos^2\varphi + \sin^2\varphi} = 1$ from $(0,0)$. **A1**

So the curve to which this normal is a tangent is a circle centre $(0,0)$, radius 1 which is thus $x^2 + y^2 = 1$ **M1 A1 (5)**