STEP III, 2021, Q1

1 **(i)** A curve has parametric equations

$$x = -4\cos^3 t\,, \quad y = 12\sin t - 4\sin^3 t\,.$$

Find the equation of the normal to this curve at the point

$$(-4\cos^3\phi\,,\ 12\sin\phi - 4\sin^3\phi),$$

where $0 < \phi < \frac{1}{2}\pi$.

Verify that this normal is a tangent to the curve

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = 4$$

at the point $(8\cos^3\phi\,,\ 8\sin^3\phi)$.

(ii) A curve has parametric equations

$$x = \cos t + t\sin t\,, \quad y = \sin t - t\cos t.$$

Find the equation of the normal to this curve at the point

$$(\cos\phi + \phi\sin\phi\,,\ \sin\phi - \phi\cos\phi),$$

where $0 < \phi < \frac{1}{2}\pi$.

Determine the perpendicular distance from the origin to this normal, and hence find the equation of a curve, independent of ϕ, to which this normal is a tangent.