STEP III, 2020, Q8 MS

8. (i) All terms of the sequence are positive integers because they are all either equal to a previous term or the sum of two previous terms which are positive integers.

Thus, for $k \geq 1$, as $u_{2k} = u_k$ and $u_{2k+1} = u_k + u_{k+1}$, $u_{2k+1} - u_{2k} = u_{k+1} \geq 1$

Also, $u_{2k+1} - u_{2k+2} = u_k + u_{k+1} - u_{k+1} = u_k \geq 1$. Thus, the required result is proved for terms from the third onwards. (The only terms not included in this proof are the first two, which are in case both equal to 1).

(ii) Suppose that $u_{2k} = c$, and that $u_{2k+1} = d$, for $k \geq 1$, where d and c share a common factor greater than one, then $u_k = c$, as $u_{2k} = u_k$, and $u_{k+1} = d - c \geq 1$ as $u_{2k+1} = u_k + u_{k+1}$ and using (i). Then as d and c share a common factor greater than one, d-c and c share a common factor greater than one. So, two earlier terms in the sequence do share the same common factor.

Likewise, suppose that $u_{2k+2} = c$, and that $u_{2k+1} = d$, for $k \geq 1$, where d and c share a common factor greater than one, then $u_{k+1} = c$ and $u_k = d - c$ giving the same result.

This is true for pairs of consecutive terms from the second term (and third) onwards. Repeating this argument, we find that it would imply that the first two terms would share a common factor greater than one, which is a contradiction. Hence any two consecutive terms are co-prime.

(iii) For $k \geq 1$, and $m \geq 1$ suppose that $u_{2k} = c$ and $u_{2k+1} = d$, and that $u_{2k+m} = c$ and $u_{2k+m+1} = d$, then as $d > c$, $2k + m$ is even, so m is even, say $2n$. Thus, $u_k = c$ and $u_{k+1} = d - c$, and $u_{k+n} = c$ and $u_{k+n+1} = d - c$. That is, an earlier pair of terms would appear consecutively.

Likewise, if $u_{2k+2} = c$ and $u_{2k+1} = d$, and that $u_{2k+m+2} = c$ and $u_{2k+m+1} = d$, the same argument applies.

So the argument can be repeated down to the first two terms, which are of course equal, and it would imply a later pair are likewise which contradicts (i).

(iv) If (a, b) does not occur, where a and b are coprime and $a > b$, then there does not exist k such that $u_{2k+1} = a$ and $u_{2k+2} = b$. Therefore there cannot exist a k such that $u_{k+1} = b$ and $u_k = a - b$, the sum of which is a, which is smaller than $a + b$.

If (a, b) does not occur, where a and b are coprime and $a < b$, then there does not exist k such that $u_{2k} = a$ and $u_{2k+1} = b$. Therefore there cannot exist a k such that $u_k = a$ and $u_{k+1} = b - a$, the sum of which is b, which is smaller than $a + b$.

(v) Suppose that there exists an ordered pair of coprime integers (a,b) which does not occur consecutively in the sequence. Then by part (iv) the pair (a-b, b) [if a>b] or (a, b-a) [if b>a] (which has a smaller sum) does not occur. Repeating this means that a coprime pair with sum <3 does not occur. The only coprime pair of integers with sum <3 is (1, 1) which are the first two terms. Contradiction and so every ordered pair of coprime integers occurs in the sequence and by (iii) only occurs once. Therefore, there exists an n, and that n is unique such that

$q = \frac{u_n}{u_{n+1}}$, for any positive rational q (which is expressed in lowest form). So the inverse of f exists.