STEP III, 2020, Q5 MS

5. $(x-y)(x^{n-1}+x^{n-2}y+\cdots+y^{n-1}) = x^n - x^{n-1}y + x^{n-1}y - x^{n-2}y + \cdots + xy^{n-1} - y^n$

$$= x^n - y^n$$

as each even numbered term cancels with its subsequent term.

(i) If

$$F(x) = \frac{1}{x^n(x-k)} = \frac{A}{x-k} + \frac{f(x)}{x^n}$$

then multiplying by $x^n(x-k)$

$$1 = Ax^n + (x-k)f(x)$$

$$x = k \Rightarrow A = \frac{1}{k^n}$$

so

$$1 = \frac{x^n}{k^n} + (x-k)f(x)$$

and

$$f(x) = \frac{1}{x-k}\left(1 - \left(\frac{x}{k}\right)^n\right)$$

as required.

Thus

$$F(x) = \frac{\frac{1}{k^n}}{x-k} + \frac{\frac{1}{x-k}\left(1-\left(\frac{x}{k}\right)^n\right)}{x^n}$$

$$= \frac{1}{k^n(x-k)} - \frac{x^n - k^n}{k^n x^n (x-k)}$$

and so, by the result of the stem,

$$F(x) = \frac{1}{k^n(x-k)} - \frac{1}{k^n x^n}\sum_{r=1}^{n} x^{n-r}k^{r-1}$$

$$= \frac{1}{k^n(x-k)} - \frac{1}{k}\sum_{r=1}^{n}\frac{1}{k^{n-r}x^r}$$

(ii)

$$x^nF(x)=\frac{1}{x-k}=\frac{x^n}{k^n(x-k)}-\frac{1}{k}\sum_{r=1}^{n}\frac{x^{n-r}}{k^{n-r}}$$

Differentiating with respect to x ,

$$\frac{-1}{(x-k)^2}=\frac{nx^{n-1}}{k^n(x-k)}-\frac{x^n}{k^n(x-k)^2}-\frac{1}{k}\sum_{r=1}^{n}\frac{(n-r)x^{n-r-1}}{k^{n-r}}$$

Multiplying by $\frac{-1}{x^n}$

$$\frac{1}{x^n(x-k)^2}=\frac{-n}{xk^n(x-k)}+\frac{1}{k^n(x-k)^2}+\sum_{r=1}^{n}\frac{n-r}{k^{n+1-r}x^{r+1}}$$

(iii)

$$\int_2^N\frac{1}{x^3(x-1)^2}dx=\int_2^N\frac{-3}{x(x-1)}+\frac{1}{(x-1)^2}+\sum_{r=1}^{3}\frac{3-r}{x^{r+1}}\,dx$$

$$=\int_2^N\frac{3}{x}-\frac{3}{(x-1)}+\frac{1}{(x-1)^2}+\sum_{r=1}^{3}\frac{3-r}{x^{r+1}}\,dx$$

$$=\left[3\ln x-3\ln(x-1)-\frac{1}{x-1}-\sum_{r=1}^{3}\frac{3-r}{rx^r}\right]_2^N$$

$$=\left[3\ln\left(\frac{x}{x-1}\right)-\frac{1}{x-1}-\sum_{r=1}^{3}\frac{3-r}{rx^r}\right]_2^N$$

$$=3\ln\left(\frac{N}{N-1}\right)-\frac{1}{N-1}-\sum_{r=1}^{3}\frac{3-r}{rN^r}-3\ln(2)+1+\sum_{r=1}^{3}\frac{3-r}{r2^r}$$

As $N\to\infty$, $\left(\frac{N-1}{N}\right)\to 1$, so $3\ln\left(\frac{N-1}{N}\right)\to 0$, and $\frac{1}{N-1}\to 0$, $\frac{1}{N^r}\to 0$

So the limit of the integral is,

$$-3\ln 2 + 1 + \frac{2}{2} + \frac{1}{8} = -3\ln 2 + \frac{17}{8}$$

Alternatives for stem using sum of GP or proof by induction