

## STEP III, 2020, Q5 MS

$$5. (x - y)(x^{n-1} + x^{n-2}y + \dots + y^{n-1}) = x^n - x^{n-1}y + x^{n-1}y - x^{n-2}y + \dots + xy^{n-1} - y^n$$

$$= x^n - y^n$$

as each even numbered term cancels with its subsequent term.

(i) If

$$F(x) = \frac{1}{x^n(x-k)} = \frac{A}{x-k} + \frac{f(x)}{x^n}$$

then multiplying by  $x^n(x-k)$

$$1 = Ax^n + (x-k)f(x)$$

$$x = k \Rightarrow A = \frac{1}{k^n}$$

so

$$1 = \frac{x^n}{k^n} + (x-k)f(x)$$

and

$$f(x) = \frac{1}{x-k} \left( 1 - \left( \frac{x}{k} \right)^n \right)$$

as required.

Thus

$$F(x) = \frac{\frac{1}{k^n}}{x-k} + \frac{\frac{1}{x-k} \left( 1 - \left( \frac{x}{k} \right)^n \right)}{x^n}$$

$$= \frac{1}{k^n(x-k)} - \frac{x^n - k^n}{k^n x^n (x-k)}$$

and so, by the result of the stem,

$$F(x) = \frac{1}{k^n(x-k)} - \frac{1}{k^n x^n} \sum_{r=1}^n x^{n-r} k^{r-1}$$

$$= \frac{1}{k^n(x-k)} - \frac{1}{k} \sum_{r=1}^n \frac{1}{k^{n-r} x^r}$$



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(ii)

$$x^n F(x) = \frac{1}{x-k} = \frac{x^n}{k^n(x-k)} - \frac{1}{k} \sum_{r=1}^n \frac{x^{n-r}}{k^{n-r}}$$

Differentiating with respect to  $x$ ,

$$\frac{-1}{(x-k)^2} = \frac{nx^{n-1}}{k^n(x-k)} - \frac{x^n}{k^n(x-k)^2} - \frac{1}{k} \sum_{r=1}^n \frac{(n-r)x^{n-r-1}}{k^{n-r}}$$

Multiplying by  $\frac{-1}{x^n}$

$$\frac{1}{x^n(x-k)^2} = \frac{-n}{xk^n(x-k)} + \frac{1}{k^n(x-k)^2} + \sum_{r=1}^n \frac{n-r}{k^{n+1-r}x^{r+1}}$$

(iii)

$$\int_2^N \frac{1}{x^3(x-1)^2} dx = \int_2^N \frac{-3}{x(x-1)} + \frac{1}{(x-1)^2} + \sum_{r=1}^3 \frac{3-r}{x^{r+1}} dx$$

$$= \int_2^N \frac{3}{x} - \frac{3}{(x-1)} + \frac{1}{(x-1)^2} + \sum_{r=1}^3 \frac{3-r}{x^{r+1}} dx$$

$$= \left[ 3 \ln x - 3 \ln(x-1) - \frac{1}{x-1} - \sum_{r=1}^3 \frac{3-r}{rx^r} \right]_2^N$$

$$= \left[ 3 \ln \left( \frac{x}{x-1} \right) - \frac{1}{x-1} - \sum_{r=1}^3 \frac{3-r}{rx^r} \right]_2^N$$

$$= 3 \ln \left( \frac{N}{N-1} \right) - \frac{1}{N-1} - \sum_{r=1}^3 \frac{3-r}{rN^r} - 3 \ln(2) + 1 + \sum_{r=1}^3 \frac{3-r}{r2^r}$$

As  $N \rightarrow \infty$ ,  $\left(\frac{N-1}{N}\right) \rightarrow 1$ , so  $3 \ln \left(\frac{N-1}{N}\right) \rightarrow 0$ , and  $\frac{1}{N-1} \rightarrow 0$ ,  $\frac{1}{N^r} \rightarrow 0$

So the limit of the integral is,



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$$-3 \ln 2 + 1 + \frac{2}{2} + \frac{1}{8} = -3 \ln 2 + \frac{17}{8}$$

Alternatives for stem using sum of GP or proof by induction



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