STEP III, 2020, Q5

5 Show that for positive integer n, $x^n - y^n = (x-y)\sum_{r=1}^{n} x^{n-r}y^{r-1}$.

(i) Let F be defined by

$$\mathrm{F}(x) = \frac{1}{x^n(x-k)} \quad \text{for } x \neq 0, k$$

where n is a positive integer and $k \neq 0$.

(a) Given that

$$\mathrm{F}(x) = \frac{A}{x-k} + \frac{\mathrm{f}(x)}{x^n},$$

where A is a constant and $\mathrm{f}(x)$ is a polynomial, show that

$$\mathrm{f}(x) = \frac{1}{x-k}\left(1 - \left(\frac{x}{k}\right)^n\right).$$

Deduce that

$$\mathrm{F}(x) = \frac{1}{k^n(x-k)} - \frac{1}{k}\sum_{r=1}^{n}\frac{1}{k^{n-r}x^r}.$$

(b) By differentiating $x^n\,\mathrm{F}(x)$, prove that

$$\frac{1}{x^n(x-k)^2} = \frac{1}{k^n(x-k)^2} - \frac{n}{xk^n(x-k)} + \sum_{r=1}^{n}\frac{n-r}{k^{n+1-r}x^{r+1}}.$$

(ii) Hence evaluate the limit of

$$\int_2^N \frac{1}{x^3(x-1)^2}\,\mathrm{d}x$$

as $N \to \infty$, justifying your answer.