STEP III 2020 Question 10 — Mark Scheme

10. If the extension in the equilibrium position is , then

$$mg = \frac{kmgd}{a}$$

Thus, $d = \dfrac{a}{k}$
If the extension when the particle is released is $d + x$, then the equation of motion is

$$m\ddot{x} = mg - \frac{kmg(d+x)}{a} = mg - \frac{kmgd}{a} - \frac{kmgx}{a} = -\frac{kmgx}{a}$$

$$\ddot{x} = -\frac{kgx}{a}$$

This is simple harmonic motion with period $\dfrac{2\pi}{\Omega}$ where $\Omega^2 = \dfrac{kg}{a}$, i.e. $kg = a\Omega^2$ as required.
Let y be the displacement of the platform below the centre point of its oscillation, then, $y = b - x$
and $\ddot{y} = -\omega^2 y = -\omega^2(b-x)$ (x newly defined as in question)
Thus, the equation of motion of the particle becomes

$$m\ddot{y} = mg - R - \frac{kmg(h-a-x)}{a}$$

So,

$$-m\omega^2(b-x) = mg - R - \frac{ma\Omega^2(h-a-x)}{a}$$

That is,

$$R = mg + m\Omega^2(a+x-h) + m\omega^2(b-x)$$

as required.
To remain in contact, $R \geq 0$ for $0 \leq x \leq 2b$

$$R = mg + m\Omega^2(a-h) + m\omega^2 b + m(\Omega^2 - \omega^2)x$$

so if $\omega < \Omega$, the minimum value of R is $mg + m\Omega^2(a-h) + m\omega^2 b$, (when $x = 0$)
thus $mg + m\Omega^2(a-h) + m\omega^2 b \geq 0$
Rearranging,

$$h \leq \frac{g + \omega^2 b}{\Omega^2} + a = \frac{a}{k} + \frac{\omega^2 b}{\Omega^2} + a = a\left(1 + \frac{1}{k}\right) + \frac{\omega^2 b}{\Omega^2}$$

as required.
If $\omega > \Omega$, minimum value of R is $mg + m\Omega^2(a-h) + m\omega^2 b + 2mb(\Omega^2 - \omega^2)$ (at $x = 2b$)
Thus,

$$mg + m\Omega^2(a-h) + m\omega^2 b + 2mb(\Omega^2 - \omega^2) \geq 0$$

and so,

$$h \leq \frac{g+\omega^2 b}{\Omega^2} + a + \frac{2b(\Omega^2-\omega^2)}{\Omega^2} = a\left(1+\frac{1}{k}\right) + \frac{\omega^2 b}{\Omega^2} + 2b - \frac{2\omega^2 b}{\Omega^2} = a\left(1+\frac{1}{k}\right) - \frac{\omega^2 b}{\Omega^2} + 2b$$

Thus, if $\omega < \Omega$, $h \leq a\left(1+\frac{1}{k}\right) + \frac{\omega^2 b}{\Omega^2} < a\left(1+\frac{1}{k}\right) + b$;
if $\omega > \Omega$, $h \leq a\left(1+\frac{1}{k}\right) - \frac{\omega^2 b}{\Omega^2} + 2b < a\left(1+\frac{1}{k}\right) + b$
If $\omega = \Omega$, then $R = mg + m\Omega^2(a-h) + m\omega^2 b \geq 0$,
so,

$$h \leq a\left(1+\frac{1}{k}\right) + b$$

Alternative

stem measuring y below A

$$m\ddot{y} = mg - \frac{kmg(y-a)}{a}$$

$$\ddot{y} = -\frac{kg}{a}\,y + kg\left(1+\frac{1}{k}\right)$$

and for platform introduced

$$m\ddot{y} = mg - R - \frac{kmg(y-a)}{a}$$