STEP III 2019 Question 11 — Mark Scheme

(i) If X is the number of customers who take sand,

$$P(X=r)=\sum_{i=r}^{\infty}\frac{\lambda^i e^{-\lambda}}{i!}\times\binom{i}{r}p^r(1-p)^{i-r}$$

M1 A1 A1

$$=\frac{(\lambda p)^r e^{-\lambda}}{r!}\sum_{I=0}^{\infty}\frac{[\lambda(1-p)]^i}{i!}$$

M1

$$=\frac{(\lambda p)^r e^{-\lambda}}{r!}e^{\lambda(1-p)}=\frac{(\lambda p)^r e^{-\lambda p}}{r!}$$

A1*

and so it follows a Poisson distribution with mean λp. [5]

(ii)

$$E(\text{mass taken})=0\cdot e^{-\lambda p}+kS\cdot\frac{(\lambda p)^1e^{-\lambda p}}{1!}+[kS+k(1-k)S]\cdot\frac{(\lambda p)^2e^{-\lambda p}}{2!}$$
$$+[kS+k(1-k)S+k(1-k)^2S]\cdot\frac{(\lambda p)^3e^{-\lambda p}}{3!}+\cdots$$

M1 A1

$$=e^{-\lambda p}kS\left[\lambda p+\frac{1-(1-k)^2}{1-(1-k)}\frac{(\lambda p)^2}{2!}+\frac{1-(1-k)^3}{1-(1-k)}\frac{(\lambda p)^3}{3!}+\cdots\right]\quad\text{for } k\neq 0$$

E1

$$=e^{-\lambda p}\frac{kS}{k}\left[\left(\lambda p+\frac{(\lambda p)^2}{2!}+\frac{(\lambda p)^3}{3!}+\cdots\right)\right.$$
$$\left.-\left((1-k)\lambda p+\frac{(1-k)^2(\lambda p)^2}{2!}+\frac{(1-k)^3(\lambda p)^3}{3!}+\cdots\right)\right]$$
$$=e^{-\lambda p}S\Big((e^{\lambda p}-1)-(e^{(1-k)\lambda p}-1)\Big)$$

M1

$$=S\left(1-e^{-k\lambda p}\right)$$

A1*

If $k = 0$, then the expected sand taken is zero which is $S(1 - e^{-k\lambda p})$ **E1** **[6]**

(iii) Using the working form part (i), the probability that r customers take sand is

$$\frac{(\lambda p)^r e^{-\lambda p}}{r!}$$

and from (ii) the mass of sand taken is

$$kS\frac{1-(1-k)^r}{1-(1-k)} = S(1-(1-k)^r)$$

so the amount that the merchant's assistant takes is $S(1-k)^r$.
The probability that the assistant takes the golden grain in this case is thus

$$\frac{kS(1-k)^r}{S} = k(1-k)^r$$

M1 A1

The probability that the assistant takes the golden grain is

$$P = \sum_{r=0}^{\infty} k(1-k)^r \frac{(\lambda p)^r e^{-\lambda p}}{r!} = ke^{-\lambda p}\sum_{r=0}^{\infty}\frac{[(1-k)\lambda p]^r}{r!} = ke^{-\lambda p}e^{(1-k)\lambda p} = ke^{-k\lambda p}$$

M1 A1 **[4]**

If $k = 0$, no sand is taken by anyone including the assistant so P should be zero, $0e^0 = 0$ **E1**
As $k \to 1$, $ke^{-k\lambda p} \to e^{-\lambda p}$ which is the probability that no customer takes sand, in the limit, the only way the assistant can take sand, in which case the grain is bound to be in his sand.. **E1**
[2]

$$P = ke^{-k\lambda p}$$

$$\frac{dP}{dk} = e^{-k\lambda p} - k\lambda p\, e^{-k\lambda p} = (1-k\lambda p)e^{-k\lambda p}$$

M1

For a maximum, $\frac{dP}{dk} = 0$, so $k = \frac{1}{\lambda p}$. (Note $p\lambda > 1$, so $\frac{1}{\lambda p} < 1$ as required because $k < 1$) **A1**
Justifying that this gives a maximum either:-
$k < \frac{1}{\lambda p} \Rightarrow \frac{dP}{dk} > 0$ and $k > \frac{1}{\lambda p} \Rightarrow \frac{dP}{dk} < 0$ as required
or

$$\frac{d^2P}{dk^2} = -\lambda p(1-k\lambda p)e^{-k\lambda p} - \lambda p\, e^{-k\lambda p} = \lambda p\, e^{-k\lambda p}[k\lambda p - 2]$$

and for $k = \frac{1}{\lambda p}$, $\frac{d^2P}{dk^2} = -\lambda p\, e^{-1} < 0$ as required. **E1** **[3]**

SCAN OR CLICK QUESTION SCAN OR CLICK EXAMINER'S COMMENTS