STEP III, 2018, Q5 MS

5. (i)

$(k+1)(\ A_{k+1} - G_{k+1}) - k(A_k - G_k) \geq 0$

$\Leftrightarrow\ \ (a_1 + a_2 + \cdots + a_k + a_{k+1}) - (k+1)G_{k+1} - (a_1 + a_2 + \cdots + a_k) + kG_k \geq 0$ **M1**

$\Leftrightarrow\ \ a_{k+1} + kG_k \geq (k+1)G_{k+1}$

$\Leftrightarrow\ \ \frac{a_{k+1}}{G_k} + k \geq (k+1)\frac{G_{k+1}}{G_k}$ as $G_k > 0$ **M1 E1**

$\frac{G_{k+1}}{G_k} = \frac{\left({G_k}^k a_{k+1}\right)^{1/k+1}}{G_k} = \left(\frac{a_{k+1}}{G_k}\right)^{1/k+1} = \lambda_k$ **B1**

So

$\frac{a_{k+1}}{G_k} + k \geq (k+1)\frac{G_{k+1}}{G_k} \Leftrightarrow {\lambda_k}^{k+1} + k \geq (k+1)\lambda_k$

$\Leftrightarrow\ {\lambda_k}^{k+1} - (k+1)\lambda_k + k \geq 0$ as required. **M1A1 (6)**

(ii) $f(x) = x^{k+1} - (k+1)x + k$

So $f'(x) = (k+1)x^k - (k+1) = (k+1)(x^k - 1)$ and $f''(x) = (k+1)kx^{k-1}$ **M1M1**

Thus, if x is positive, there is a single stationary point for $x = 1$ and it is a minimum. **E1**

$f(1) = 0$

and so $f(x) = x^{k+1} - (k+1)x + k \geq 0$ **E1* (4)**

(iii) (a) Assume $A_k \geq G_k$ for some particular k **B1**

then by (ii), the condition for (i) is met and so $A_{k+1} - G_{k+1} \geq \frac{k}{k+1}(A_k - G_k)$ **E1**

and thus $A_{k+1} \geq G_{k+1}$

$A_1 = a_1$ and $G_1 = a_1$ and so $A_1 \geq G_1$ (in fact $A_1 = G_1$) **B1**

Thus, by the principle of mathematical induction, $A_n \geq G_n$ for all n **B1 (4)**

(b) If $A_k = G_k$ for some k, then as $A_n \geq G_n$ for all n, $A_{k-1} \geq G_{k-1}$ **E1**

and by (i) and (ii) $A_{k-1} = G_{k-1}$ and **E1**

$$\left(\frac{a_k}{G_{k-1}}\right)^{1/k} = 1$$

in which case $a_k = G_{k-1}$. **B1**

But as $A_n = G_n$, $A_k = G_k$ and $a_k = G_{k-1}$ for all k, for $k = 1$ to n E1

But, $A_1 = G_1 = a_1$ B1 and so $a_2 = G_1 = a_1$ and thus $A_2 = G_2 = a_1$ and $a_3 = G_2 = a_1$ and so on up to $A_n = G_n = a_1$

Hence $a_1 = a_2 = a_3 = \cdots = a_n$ E1 (6)