STEP III, 2018, Q5

5 The real numbers a_1, a_2, a_3, ... are all positive. For each positive integer n, A_n and G_n are defined by

$$A_n = \frac{a_1 + a_2 + \cdots + a_n}{n} \qquad \text{and} \qquad G_n = \left(a_1 a_2 \cdots a_n\right)^{1/n}.$$

(i) Show that, for any given positive integer k,

$$(k+1)(A_{k+1} - G_{k+1}) \geqslant k(A_k - G_k)$$

if and only if

$$\lambda_k^{k+1} - (k+1)\lambda_k + k \geqslant 0\,,$$

where $\lambda_k = \left(\dfrac{a_{k+1}}{G_k}\right)^{\frac{1}{k+1}}$.

(ii) Let

$$\mathrm{f}(x) = x^{k+1} - (k+1)x + k\,,$$

where $x > 0$ and k is a positive integer. Show that $\mathrm{f}(x) \geqslant 0$ and that $\mathrm{f}(x) = 0$ if and only if $x = 1$.

(iii) Deduce that:

(a) $A_n \geqslant G_n$ for all n;

(b) if $A_n = G_n$ for some n, then $a_1 = a_2 = \cdots = a_n$.