

## STEP III, 2018, Q4 MS

4.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\frac{2x}{a^2} - \frac{2y}{b^2} \frac{dy}{dx} = 0$$

**M1**

(Alternatively,

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{b \sec^2 \theta}{a \sec \theta \tan \theta}$$

also earns **M1**)

Thus at P

$$\frac{dy}{dx} = \frac{a \sec \theta}{a^2} \frac{b^2}{b \tan \theta} = \frac{b}{a \sin \theta}$$

**A1**

So the tangent at P is

$$y - b \tan \theta = \frac{b}{a \sin \theta} (x - a \sec \theta)$$

**M1**

Hence

$$ay \sin \theta - ab \frac{\sin^2 \theta}{\cos \theta} = bx - ab \frac{1}{\cos \theta}$$

So

$$bx - ay \sin \theta = ab \left( \frac{1}{\cos \theta} - \frac{\sin^2 \theta}{\cos \theta} \right) = ab \frac{\cos^2 \theta}{\cos \theta} = ab \cos \theta$$

as required.

**A1\* (4)**

(i) S is the intersection of  $bx - ay \sin \theta = ab \cos \theta$  and  $\frac{x}{a} = \frac{y}{b}$

So  $bx - bx \sin \theta = ab \cos \theta$  **M1**

Thus, S is  $\left( \frac{a \cos \theta}{1 - \sin \theta}, \frac{b \cos \theta}{1 - \sin \theta} \right)$  **A1**

Similarly, T is the intersection of  $bx - ay \sin \theta = ab \cos \theta$  and  $\frac{x}{a} = -\frac{y}{b}$

So T is  $\left( \frac{a \cos \theta}{1 + \sin \theta}, -\frac{b \cos \theta}{1 + \sin \theta} \right)$  **M1A1**

The midpoint of ST is therefore  $\left( \frac{1}{2} \left( \frac{a \cos \theta}{1 - \sin \theta} + \frac{a \cos \theta}{1 + \sin \theta} \right), \frac{1}{2} \left( \frac{b \cos \theta}{1 - \sin \theta} - \frac{b \cos \theta}{1 + \sin \theta} \right) \right)$  **M1**

$$\frac{1}{2} \left( \frac{a \cos \theta}{1 - \sin \theta} + \frac{a \cos \theta}{1 + \sin \theta} \right) = \frac{1}{2} a \cos \theta \frac{1 + \sin \theta + 1 - \sin \theta}{(1 - \sin \theta)(1 + \sin \theta)} = a \sec \theta$$



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$$\frac{1}{2} \left( \frac{b \cos \theta}{1 - \sin \theta} - \frac{b \cos \theta}{1 + \sin \theta} \right) = \frac{1}{2} b \cos \theta \frac{1 + \sin \theta - 1 + \sin \theta}{(1 - \sin \theta)(1 + \sin \theta)} = b \tan \theta$$

M1

which means it is P. **A1 (7)**

(ii) As the tangents at P and Q are perpendicular,

$$\frac{b}{a \sin \theta} \times \frac{b}{a \sin \varphi} = -1$$

B1

(This is possible because  $a > b$ )

That is

$$a^2 \sin \theta \sin \varphi + b^2 = 0$$

The intersection of the tangents is given by the solution of

$$bx - ay \sin \theta = ab \cos \theta$$

$$bx - ay \sin \varphi = ab \cos \varphi$$

Thus

$$x = a \frac{(\sin \varphi \cos \theta - \sin \theta \cos \varphi)}{(\sin \varphi - \sin \theta)}$$

M1

and

$$y = b \frac{(\cos \theta - \cos \varphi)}{(\sin \varphi - \sin \theta)}$$

$$x^2 = \left[ a \frac{(\sin \varphi \cos \theta - \sin \theta \cos \varphi)}{(\sin \varphi - \sin \theta)} \right]^2$$

A1

$$y^2 = -a^2 \sin \theta \sin \varphi \left[ \frac{(\cos \theta - \cos \varphi)}{(\sin \varphi - \sin \theta)} \right]^2$$

A1

Note  $\sin \theta \sin \varphi = -\frac{b^2}{a^2} < 0$  so  $\sin \varphi \neq \sin \theta$  and so  $\sin \varphi - \sin \theta \neq 0$  **E1**

So

$$\begin{aligned} x^2 + y^2 &= \frac{a^2}{(\sin \varphi - \sin \theta)^2} [(\sin \varphi \cos \theta - \sin \theta \cos \varphi)^2 - \sin \theta \sin \varphi (\cos \theta - \cos \varphi)^2] \\ &= \frac{a^2}{(\sin \varphi - \sin \theta)^2} [\sin^2 \varphi \cos^2 \theta + \sin^2 \theta \cos^2 \varphi - \sin \theta \sin \varphi \cos^2 \theta - \sin \theta \sin \varphi \cos^2 \varphi] \end{aligned}$$

M1



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$$\begin{aligned} &= \frac{a^2}{(\sin \varphi - \sin \theta)^2} [\sin^2 \varphi (1 - \sin^2 \theta) + \sin^2 \theta (1 - \sin^2 \varphi) - 2 \sin \varphi \sin \theta + 2 \sin \varphi \sin \theta - \\ &\quad \sin \theta \sin \varphi \cos^2 \theta - \sin \theta \sin \varphi \cos^2 \varphi] \quad \mathbf{M1} \\ &= \frac{a^2}{(\sin \varphi - \sin \theta)^2} [(\sin \varphi - \sin \theta)^2 + \sin \varphi \sin \theta (2 - 2 \sin \varphi \sin \theta - \cos^2 \theta - \cos^2 \varphi)] \\ &= \frac{a^2}{(\sin \varphi - \sin \theta)^2} [(\sin \varphi - \sin \theta)^2 + \sin \varphi \sin \theta (\sin^2 \varphi - 2 \sin \varphi \sin \theta + \sin^2 \theta)] \\ &= a^2 + a^2 \sin \theta \sin \varphi = a^2 - b^2 \end{aligned}$$

as required.

**M1A1\* (9)**



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