STEP III, 2018, Q4 MS

4.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0$$

M1

(Alternatively,

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{b\sec^2\theta}{a\sec\theta\tan\theta}$$

also earns M1)

Thus at P

$$\frac{dy}{dx} = \frac{a\sec\theta}{a^2}\frac{b^2}{b\tan\theta} = \frac{b}{a\sin\theta}$$

A1

So the tangent at P is

$$y - b\tan\theta = \frac{b}{a\sin\theta}(x - a\sec\theta)$$

M1

Hence

$$ay\sin\theta - ab\frac{\sin^2\theta}{\cos\theta} = bx - ab\frac{1}{\cos\theta}$$

So

$$bx - ay\sin\theta = ab\left(\frac{1}{\cos\theta} - \frac{\sin^2\theta}{\cos\theta}\right) = ab\frac{\cos^2\theta}{\cos\theta} = ab\cos\theta$$

as required. A1* (4)

(i) S is the intersection of $bx - ay\sin\theta = ab\cos\theta$ and $\frac{x}{a} = \frac{y}{b}$

So $bx - bx\sin\theta = ab\cos\theta$ M1

Thus, S is $\left(\frac{a\cos\theta}{1-\sin\theta}, \frac{b\cos\theta}{1-\sin\theta}\right)$ A1

Similarly, T is the intersection of $bx - ay\sin\theta = ab\cos\theta$ and $\frac{x}{a} = -\frac{y}{b}$

So T is $\left(\frac{a\cos\theta}{1+\sin\theta}, -\frac{b\cos\theta}{1+\sin\theta}\right)$ M1A1

The midpoint of ST is therefore $\left(\frac{1}{2}\left(\frac{a\cos\theta}{1-\sin\theta} + \frac{a\cos\theta}{1+\sin\theta}\right), \frac{1}{2}\left(\frac{b\cos\theta}{1-\sin\theta} - \frac{b\cos\theta}{1+\sin\theta}\right)\right)$ M1

$$\frac{1}{2}\left(\frac{a\cos\theta}{1-\sin\theta} + \frac{a\cos\theta}{1+\sin\theta}\right) = \frac{1}{2}a\cos\theta\frac{1+\sin\theta+1-\sin\theta}{(1-\sin\theta)(1+\sin\theta)} = a\sec\theta$$

$$\frac{1}{2}\left(\frac{b\cos\theta}{1-\sin\theta}-\frac{b\cos\theta}{1+\sin\theta}\right)=\frac{1}{2}b\cos\theta\frac{1+\sin\theta-1+\sin\theta}{(1-\sin\theta)(1+\sin\theta)}=b\tan\theta$$

M1

which means it is P. A1 (7)

(ii) As the tangents at P and Q are perpendicular,

$$\frac{b}{a\sin\theta}\times\frac{b}{a\sin\varphi}=-1$$

B1

(This is possible because $a > b$)

That is

$$a^2\sin\theta\ \sin\varphi+b^2=0$$

The intersection of the tangents is given by the solution of

$$bx-ay\sin\theta=ab\cos\theta$$

$$bx-ay\sin\varphi=ab\cos\varphi$$

Thus

$$x=a\frac{(\sin\varphi\cos\theta-\sin\theta\cos\varphi)}{(\sin\varphi-\sin\theta)}$$

M1

and

$$y=b\frac{(\cos\theta-\cos\varphi)}{(\sin\varphi-\sin\theta)}$$

$$x^2=\left[a\frac{(\sin\varphi\cos\theta-\sin\theta\cos\varphi)}{(\sin\varphi-\sin\theta)}\right]^2$$

A1

$$y^2=-a^2\sin\theta\ \sin\varphi\left[\frac{(\cos\theta-\cos\varphi)}{(\sin\varphi-\sin\theta)}\right]^2$$

A1

Note $\sin\theta\ \sin\varphi=-\frac{b^2}{a^2}<0$ so $\sin\varphi\neq\sin\theta$ and so $\sin\varphi-\sin\theta\neq0$ E1

So

$$x^2+y^2=\frac{a^2}{(\sin\varphi-\sin\theta)^2}[(\sin\varphi\cos\theta-\sin\theta\cos\varphi)^2-\sin\theta\ \sin\varphi\,(\cos\theta-\cos\varphi)^2]$$

$$=\frac{a^2}{(\sin\varphi-\sin\theta)^2}[\sin^2\varphi\cos^2\theta+\sin^2\theta\cos^2\varphi-\sin\theta\ \sin\varphi\cos^2\theta-\sin\theta\ \sin\varphi\ \cos^2\varphi]$$

M1

$$= \frac{a^2}{(\sin\varphi - \sin\theta)^2}[\sin^2\varphi\,(1-\sin^2\theta) + \sin^2\theta\,(1-\sin^2\varphi) - 2\sin\varphi\sin\theta + 2\sin\varphi\sin\theta -$$
$$\sin\theta\ \sin\varphi\cos^2\theta - \sin\theta\ \sin\varphi\ \cos^2\varphi]$$ M1

$$= \frac{a^2}{(\sin\varphi - \sin\theta)^2}[(\sin\varphi - \sin\theta)^2 + \sin\varphi\sin\theta(2 - 2\sin\varphi\sin\theta - \cos^2\theta - \cos^2\varphi)]$$

$$= \frac{a^2}{(\sin\varphi - \sin\theta)^2}[(\sin\varphi - \sin\theta)^2 + \sin\varphi\sin\theta\ (\sin^2\varphi - 2\sin\varphi\sin\theta + \sin^2\theta)]$$

$$= a^2 + a^2\sin\theta\ \sin\varphi = a^2 - b^2$$

as required. M1A1* (9)