STEP III, 2018, Q4

4 The point $P(a\sec\theta, b\tan\theta)$ lies on the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\,,$$

where $a > b > 0$. Show that the equation of the tangent to the hyperbola at P can be written as

$$bx - ay\sin\theta = ab\cos\theta\,.$$

(i) This tangent meets the lines $\dfrac{x}{a} = \dfrac{y}{b}$ and $\dfrac{x}{a} = -\dfrac{y}{b}$ at S and T, respectively.

How is the mid-point of ST related to P?

(ii) The point $Q(a\sec\phi, b\tan\phi)$ also lies on the hyperbola and the tangents to the hyperbola at P and Q are perpendicular. These two tangents intersect at (x, y).

Obtain expressions for x^2 and y^2 in terms of a, θ and ϕ.

Hence, or otherwise, show that $x^2 + y^2 = a^2 - b^2$.