

STEP III, 2018, Q3 MS

3.

$$\begin{aligned}
 x^a(x^b(x^c y)')' &= x^a[x^b(x^c y' + cx^{c-1}y)]' \\
 & \qquad \qquad \qquad \text{M1} \\
 &= x^a[x^{b+c}y' + cx^{b+c-1}y]' \\
 &= x^a[x^{b+c}y'' + cx^{b+c-1}y' + (b+c)x^{b+c-1}y' + c(b+c-1)x^{b+c-2}y] \\
 &= x^{a+b+c}y'' + (b+2c)x^{a+b+c-1}y' + c(b+c-1)x^{a+b+c-2}y \\
 & \qquad \qquad \qquad \text{M1A1}
 \end{aligned}$$

This is of the required form if

$$\begin{aligned}
 a + b + c &= 2 \\
 b + 2c &= 1 - 2p \\
 c(b + c - 1) &= p^2 - q^2 \\
 & \qquad \qquad \qquad \text{M1}
 \end{aligned}$$

Thus

$$\begin{aligned}
 c(1 - 2p - 2c + c - 1) &= p^2 - q^2 \\
 c^2 + 2pc + p^2 &= q^2 \\
 c + p &= \pm q \\
 & \qquad \qquad \qquad \text{M1}
 \end{aligned}$$

Thus it is possible if

$$c = q - p, \quad b = 1 - 2q, \quad a = 1 + p + q$$

or

$$c = -q - p, \quad b = 1 + 2q, \quad a = 1 + p - q$$

A1 (6)

(i) $x^2 y'' + (1 - 2p)xy' + (p^2 - q^2)y = 0$

So

$$\begin{aligned}
 x^a(x^b(x^c y)')' &= 0 \\
 (x^b(x^c y)')' &= 0 \\
 x^b(x^c y)' &= A \\
 & \qquad \qquad \qquad \text{M1} \\
 (x^c y)' &= Ax^{-b}
 \end{aligned}$$

$$x^c y = \frac{Ax^{-b+1}}{-b+1} + B \quad \text{unless } b = 1 \qquad \qquad \qquad \text{B1}$$



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$b \neq 1 \Rightarrow q \neq 0$ in which case

$$y = \frac{Ax^{-b-c+1}}{-b+1} + Bx^{-c}$$

$$y = \frac{Ax^{p \pm q}}{\pm 2q} + Bx^{p \pm q}$$

That is

$$y = Cx^{p+q} + Dx^{p-q}$$

M1A1

However, if $b = 1$, $x^c y = A \ln x + B$

M1

so $y = Ax^{-c} \ln x + Bx^{-c}$

So if $q = 0$, $y = Ax^p \ln x + Bx^p$

M1A1 (7)

(ii) $x^2 y'' + (1 - 2p)xy' + p^2 y = x^n$

Thus $q = 0$, and $c = -p$, $b = 1$, $a = 1 + p$

B1

$$x^a(x(x^c y)')' = x^n$$

$$(x(x^c y)')' = x^{n-a}$$

$$x(x^c y)' = \frac{x^{n+1-a}}{n+1-a} + A \text{ for } n+1-a \neq 0 \text{ or } x(x^c y)' = \ln x + A \text{ for } n+1-a = 0 \quad \mathbf{M1 B1}$$

$$n+1-a=0 \Rightarrow n=p$$

$$\text{Thus } (x^c y)' = \frac{x^{n-a}}{n+1-a} + Ax^{-1} \text{ or } (x^c y)' = x^{-1} \ln x + Ax^{-1}$$

$$\text{So } x^c y = \frac{x^{n+1-a}}{(n+1-a)^2} + A \ln x + B \text{ or } x^c y = \frac{(\ln x)^2}{2} + A \ln x + B \quad \mathbf{M1 M1}$$

$$\text{So for } n \neq p, y = \frac{x^n}{(n-p)^2} + Ax^p \ln x + Bx^p \quad \mathbf{A1}$$

$$\text{and for } n = p, y = x^n \frac{(\ln x)^2}{2} + Ax^n \ln x + Bx^n \quad \mathbf{A1 (7)}$$



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