STEP III, 2018, Q3 MS

3.

$$x^a\left(x^b(x^cy)'\right)' = x^a[x^b(x^cy' + cx^{c-1}y)]'$$

M1

$$= x^a[x^{b+c}y' + cx^{b+c-1}y]'$$

$$= x^a[x^{b+c}y'' + cx^{b+c-1}y' + (b+c)x^{b+c-1}y' + c(b+c-1)x^{b+c-2}y]$$

$$= x^{a+b+c}y'' + (b+2c)x^{a+b+c-1}y' + c(b+c-1)x^{a+b+c-2}y$$

M1A1

This is of the required form if

$$a+b+c=2$$

$$b+2c=1-2p$$

$$c(b+c-1)=p^2-q^2$$

M1

Thus

$$c(1-2p-2c+c-1)=p^2-q^2$$

$$c^2+2pc+p^2=q^2$$

$$c+p=\pm q$$

M1

Thus it is possible if

$c=q-p$, $b=1-2q$, $a=1+p+q$

or

$c=-q-p$, $b=1+2q$, $a=1+p-q$

A1 (6)

(i) $x^2y''+(1-2p)xy'+(p^2-q^2)y=0$

So

$$x^a\left(x^b(x^cy)'\right)'=0$$

$$\left(x^b(x^cy)'\right)'=0$$

$$x^b(x^cy)'=A$$

M1

$$(x^cy)'=Ax^{-b}$$

$x^cy=\frac{Ax^{-b+1}}{-b+1}+B$ unless $b=1$ **B1**

$$y = \frac{Ax^{-b-c+1}}{-b+1} + Bx^{-c}$$

$b \neq 1 \Rightarrow q \neq 0$ in which case

$$y = \frac{Ax^{p \pm q}}{\pm 2q} + Bx^{p \pm q}$$

That is

$$y = Cx^{p+q} + Dx^{p-q}$$

M1A1

However, if $b = 1$, $x^c y = A\ln x + B$ M1

so $y = Ax^{-c}\ln x + Bx^{-c}$

So if $q = 0$, $y = Ax^p \ln x + Bx^p$ M1A1 (7)

(ii) $x^2y'' + (1-2p)xy' + p^2y = x^n$

Thus $q = 0$, and $c = -p$, $b = 1$, $a = 1 + p$ B1

$$x^a(x(x^c y)')' = x^n$$

$$(x(x^c y)')' = x^{n-a}$$

$x(x^c y)' = \frac{x^{n+1-a}}{n+1-a} + A$ for $n+1-a \neq 0$ or $x(x^c y)' = \ln x + A$ for $n+1-a = 0$ M1 B1

$$n + 1 - a = 0 \Rightarrow n = p$$

Thus $(x^c y)' = \frac{x^{n-a}}{n+1-a} + Ax^{-1}$ or $(x^c y)' = x^{-1}\ln x + Ax^{-1}$

So $x^c y = \frac{x^{n+1-a}}{(n+1-a)^2} + A\ln x + B$ or $x^c y = \frac{(\ln x)^2}{2} + A\ln x + B$ M1 M1

So for $n \neq p$, $y = \frac{x^n}{(n-p)^2} + Ax^p \ln x + Bx^p$ A1

and for $n = p$, $y = x^n \frac{(\ln x)^2}{2} + Ax^n \ln x + Bx^n$ A1 (7)