STEP III, 2018, Q3

3 Show that the second-order differential equation

$$x^2y'' + (1-2p)x\,y' + (p^2-q^2)\,y = \mathrm{f}(x)\,,$$

where p and q are constants, can be written in the form

$$x^a\left(x^b(x^cy)'\right)' = \mathrm{f}(x)\,, \qquad (*)$$

where a, b and c are constants.

(i) Use $(*)$ to derive the general solution of the equation

$$x^2y'' + (1-2p)xy' + (p^2-q^2)y = 0$$

in the different cases that arise according to the values of p and q.

(ii) Use $(*)$ to derive the general solution of the equation

$$x^2y'' + (1-2p)xy' + p^2y = x^n$$

in the different cases that arise according to the values of p and n.