

STEP III, 2018, Q2 MS

2. (i)

$$\begin{aligned} \frac{dy_n}{dx} &= \frac{d}{dx} \left((-1)^n \frac{1}{z} \frac{d^n z}{dx^n} \right) = (-1)^n \left[2xe^{x^2} \frac{d^n z}{dx^n} + \frac{1}{z} \frac{d^{n+1} z}{dx^{n+1}} \right] && \text{M1} \\ &= 2x(-1)^n \frac{1}{z} \frac{d^n z}{dx^n} - (-1)^{n+1} \frac{1}{z} \frac{d^{n+1} z}{dx^{n+1}} && \text{M1} \\ &= 2xy_n - y_{n+1} \end{aligned}$$

as required.

A1* (3)

(ii) Suppose $y_{k+1} = 2xy_k - 2ky_{k-1}$ for some k **B1**

$$\frac{dy_{k+1}}{dx} = 2x \frac{dy_k}{dx} + 2y_k - 2k \frac{dy_{k-1}}{dx} \quad \text{M1}$$

So using (i)

$$\begin{aligned} 2xy_{k+1} - y_{k+2} &= 2x(2xy_k - y_{k+1}) + 2y_k - 2k(2xy_{k-1} - y_k) && \text{M1} \\ &= 4x^2y_k - 2xy_{k+1} + 2y_k + 2ky_k - 2x(2xy_{k-1} - y_{k+1}) && \text{M1} \\ &= 4x^2y_k - 2xy_{k+1} + 2y_k + 2ky_k - 4x^2y_{k-1} + 2xy_{k+1} \\ &= 2y_k + 2ky_k \end{aligned}$$

Thus $y_{k+2} = 2xy_{k+1} - 2(k+1)y_k$ which is the required result for $k+1$ **A1**

$$y_0 = 1$$

$$y_1 = (-1) \frac{1}{z} \frac{dz}{dx} = -1e^{x^2} \frac{d}{dx}(e^{-x^2}) = -e^{x^2} \cdot -2xe^{-x^2} = 2x \quad \text{B1}$$

$$\begin{aligned} y_2 &= (-1)^2 \frac{1}{z} \frac{d^2 z}{dx^2} = e^{x^2} \frac{d^2(e^{-x^2})}{dx^2} = e^{x^2} \left[\frac{d}{dx}(-2xe^{-x^2}) \right] = e^{x^2}(-2e^{-x^2} + 4x^2e^{-x^2}) \\ &= -2 + 4x^2 \end{aligned}$$

M1A1

$2xy_1 - 2 \times 1y_0 = 2x(2x) - 2 = 4x^2 - 2$ so result true for $n = 1$ **A1**

Hence $y_{n+1} = 2xy_n - 2ny_{n-1}$ for $n \geq 1$ by induction. **A1 (10)**



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As $y_{n+1} = 2xy_n - 2ny_{n-1}$, $y_{n+2} = 2xy_{n+1} - 2(n+1)y_n$ **M1**

Eliminating x ,

$$(y_{n+1})^2 - y_n y_{n+2} = 2(n+1)(y_n)^2 - 2ny_{n-1}y_{n+1}$$

M1

Thus

$$y_{n+1}(y_{n+1} + 2ny_{n-1}) = y_n(y_{n+2} + 2(n+1)y_n)$$

So

$$y_{n+1}^2 - y_n y_{n+2} = 2n(y_n^2 - y_{n-1}y_{n+1}) + 2y_n^2$$

A1 (3)

(iii) Suppose $y_k^2 - y_{k-1}y_{k+1} > 0$ for some $k \geq 1$ **B1**

Then, as $2y_k^2 \geq 0$, $2k(y_k^2 - y_{k-1}y_{k+1}) + 2y_k^2 > 0$, i.e. by (ii) $y_{k+1}^2 - y_k y_{k+2} > 0$ **E1**

Consider $k = 1$

$$y_1^2 - y_0 y_2 = (2x)^2 - 1 \times (-2 + 4x^2) = 4x^2 + 2 - 4x^2 = 2 > 0$$

B1

Hence the result $y_n^2 - y_{n-1}y_{n+1} > 0$ for $n \geq 1$

B1 (4)



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