STEP III, 2018, Q2 MS

2. (i)

$$\frac{dy_n}{dx} = \frac{d}{dx}\left((-1)^n \frac{1}{z}\frac{d^n z}{dx^n}\right) = (-1)^n \left[2xe^{x^2}\frac{d^n z}{dx^n} + \frac{1}{z}\frac{d^{n+1}z}{dx^{n+1}}\right]$$

M1

$$= 2x(-1)^n \frac{1}{z}\frac{d^n z}{dx^n} - (-1)^{n+1}\frac{1}{z}\frac{d^{n+1}z}{dx^{n+1}}$$

M1

$$= 2xy_n - y_{n+1}$$

as required. **A1* (3)**

(ii) Suppose $y_{k+1} = 2xy_k - 2ky_{k-1}$ for some k **B1**

$$\frac{dy_{k+1}}{dx} = 2x\frac{dy_k}{dx} + 2y_k - 2k\frac{dy_{k-1}}{dx}$$

M1

So using (i)

$$2xy_{k+1} - y_{k+2} = 2x(2xy_k - y_{k+1}) + 2y_k - 2k(2xy_{k-1} - y_k)$$

M1

$$= 4x^2y_k - 2xy_{k+1} + 2y_k + 2ky_k - 2x(2xy_k - y_{k+1})$$

M1

$$= 4x^2y_k - 2xy_{k+1} + 2y_k + 2ky_k - 4x^2y_k + 2xy_{k+1}$$

$$= 2y_k + 2ky_k$$

Thus $y_{k+2} = 2xy_{k+1} - 2(k+1)y_k$ which is the required result for $k+1$ **A1**

$$y_0 = 1$$

$$y_1 = (-1)\frac{1}{z}\frac{dz}{dx} = -1e^{x^2}\frac{d}{dx}\left(e^{-x^2}\right) = -e^{x^2}.-2xe^{-x^2} = 2x$$

B1

$$y_2 = (-1)^2\frac{1}{z}\frac{d^2z}{dx^2} = e^{x^2}\frac{d^2\left(e^{-x^2}\right)}{dx^2} = e^{x^2}\left[\frac{d}{dx}\left(-2xe^{-x^2}\right)\right] = e^{x^2}\left(-2e^{-x^2} + 4x^2e^{-x^2}\right)$$

$$= -2 + 4x^2$$

M1A1

$2xy_1 - 2 \times 1y_0 = 2x(2x) - 2 = 4x^2 - 2$ so result true for $n = 1$ **A1**

Hence $y_{n+1} = 2xy_n - 2ny_{n-1}$ for $n \geq 1$ by induction. **A1 (10)**

As $y_{n+1} = 2xy_n - 2ny_{n-1}$, $y_{n+2} = 2xy_{n+1} - 2(n+1)y_n$ **M1**

Eliminating x,

$$(y_{n+1})^2 - y_n y_{n+2} = 2(n+1)(y_n)^2 - 2ny_{n-1}y_{n+1}$$

M1

Thus

$$y_{n+1}(y_{n+1} + 2ny_{n-1}) = y_n(y_{n+2} + 2(n+1)y_n)$$

So

$$y_{n+1}{}^2 - y_n y_{n+2} = 2n(y_n{}^2 - y_{n-1}y_{n+1}) + 2y_n{}^2$$

A1 (3)

(iii) Suppose $y_k{}^2 - y_{k-1}y_{k+1} > 0$ for some $k \geq 1$ **B1**

Then, as $2y_k{}^2 \geq 0$, $2k(y_k{}^2 - y_{k-1}y_{k+1}) + 2y_k{}^2 > 0$, i.e. by (ii) $y_{k+1}{}^2 - y_k y_{k+2} > 0$ **E1**

Consider $k = 1$

$$y_1{}^2 - y_0 y_2 = (2x)^2 - 1 \times (-2 + 4x^2) = 4x^2 + 2 - 4x^2 = 2 > 0$$

B1

Hence the result $y_n{}^2 - y_{n-1}y_{n+1} > 0$ for $n \geq 1$

B1 (4)