

STEP III, 2017, Q8 MS

Question 8

The stem can be achieved by adding the two summations, expanding the brackets, and observing that the resulting two summed terms telescope. (i) is simply a case of using the given expression for b_m , and letting $a_m = 1$ (or any constant) in the stem result, simplifying the left-hand side using the given note, and dividing through by $\sin \frac{1}{2}x$. Part (ii) can be obtained by letting $a_m = m$ and $b_m = \sin(m-1)x - \sin mx$ (or similarly $b_m = \cos\left(m - \frac{1}{2}\right)x$) in the stem which simplifies to give $p = -\frac{1}{4}n$ and $q = \frac{1}{4}(n+1)$.

$$\begin{aligned} \sum_{m=1}^n a_m(b_{m+1} - b_m) + \sum_{m=1}^n b_{m+1}(a_{m+1} - a_m) \\ = \sum_{m=1}^n (a_m b_{m+1} - a_m b_m + b_{m+1} a_{m+1} - b_{m+1} a_m) \end{aligned}$$

M1

$$= \sum_{m=1}^n (-a_m b_m + b_{m+1} a_{m+1}) = a_{n+1} b_{n+1} - a_1 b_1$$

M1

Hence,

$$\sum_{m=1}^n a_m(b_{m+1} - b_m) = a_{n+1} b_{n+1} - a_1 b_1 - \sum_{m=1}^n b_{m+1}(a_{m+1} - a_m)$$

A1* (3)

(i) Let $a_m = 1$ (or any constant) and $b_m = \sin mx$, M1

then

$$\sum_{m=1}^n (\sin(m+1)x - \sin mx) = \sin(n+1)x - \sin x - \sum_{m=1}^n \sin(m+1)x (1-1)$$

M1 A1

So

$$\sum_{m=1}^n 2 \cos\left(m + \frac{1}{2}\right)x \sin \frac{1}{2}x = (\sin(n+1)x - \sin x)$$

M1 A1



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and therefore

$$\sum_{m=1}^n \cos\left(m + \frac{1}{2}\right)x = \frac{1}{2}(\sin(n+1)x - \sin x) \csc \frac{1}{2}x$$

A1* (6)

(ii) Let $a_m = m$ and $b_m = \sin(m-1)x - \sin mx$, **M1**

then

$$\begin{aligned} b_{m+1} - b_m &= (\sin mx - \sin(m+1)x) - (\sin(m-1)x - \sin mx) \\ &= -2 \cos\left(m + \frac{1}{2}\right)x \sin \frac{1}{2}x + 2 \cos\left(m - \frac{1}{2}\right)x \sin \frac{1}{2}x \end{aligned}$$

M1 A1

$$= 4 \sin mx \sin \frac{1}{2}x \sin \frac{1}{2}x \quad \text{M1 A1}$$

Thus, using the stem

$$\begin{aligned} \sum_{m=1}^n m \times 4 \sin mx \sin^2 \frac{1}{2}x &= (n+1)(\sin nx - \sin(n+1)x) - 1 \times (\sin(0 \times x) - \sin x) \\ &\quad - \sum_{m=1}^n (\sin mx - \sin(m+1)x) \end{aligned}$$

M1 A1

So

$$4 \sin^2 \frac{1}{2}x \sum_{m=1}^n m \sin mx = (n+1)(\sin nx - \sin(n+1)x) + \sin x - \sin x + \sin(n+1)x$$

M1 A1

$$4 \sin^2 \frac{1}{2}x \sum_{m=1}^n m \sin mx = (n+1) \sin nx - n \sin(n+1)x$$



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Thus

$$\sum_{m=1}^n m \sin mx = (p \sin nx + q \sin(n+1)x) \csc^2 \frac{1}{2}x$$

where

$$p = -\frac{1}{4}n$$

A1

and

$$q = \frac{1}{4}(n+1)$$

A1 (11)

Alternatively, let $a_m = m$ and $b_m = \cos\left(m - \frac{1}{2}\right)x$, using stem, **M1**

$$\begin{aligned} \sum_{m=1}^n m \left(\cos\left(m + \frac{1}{2}\right)x - \cos\left(m - \frac{1}{2}\right)x \right) \\ = (n+1) \cos\left(n + \frac{1}{2}\right)x - \cos \frac{1}{2}x - \sum_{m=1}^n \cos\left(m + \frac{1}{2}\right)x \end{aligned}$$

M1 A1

So,

$$\begin{aligned} \sum_{m=1}^n -2m \sin mx \sin \frac{1}{2}x \\ = (n+1) \cos\left(n + \frac{1}{2}\right)x - \cos \frac{1}{2}x - \frac{1}{2}(\sin(n+1)x - \sin x) \csc \frac{1}{2}x \end{aligned}$$

M1 A1

$$= \csc \frac{1}{2}x \left((n+1) \cos\left(n + \frac{1}{2}\right)x \sin \frac{1}{2}x - \sin \frac{1}{2}x \cos \frac{1}{2}x - \frac{1}{2}(\sin(n+1)x - \sin x) \right)$$

M1 A1

$$\begin{aligned} = \frac{1}{2} \csc \frac{1}{2}x \left(2(n+1) \cos\left(n + \frac{1}{2}\right)x \sin \frac{1}{2}x - 2 \sin \frac{1}{2}x \cos \frac{1}{2}x - (\sin(n+1)x - \sin x) \right) \\ = \frac{1}{2} \csc \frac{1}{2}x \left((n+1)(\sin(n+1)x - \sin nx) - \sin x - \sin(n+1)x + \sin x \right) \end{aligned}$$

M1 A1

$$= \frac{1}{2} \csc \frac{1}{2}x (n \sin(n+1)x - (n+1) \sin nx)$$

giving result as before.



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