STEP III, 2017, Q8 MS

Question 8

The stem can be achieved by adding the two summations, expanding the brackets, and observing that the resulting two summed terms telescope. (i) is simply a case of using the given expression for b_m, and letting $a_m = 1$ (or any constant) in the stem result, simplifying the left-hand side using the given note, and dividing through by $\sin\frac{1}{2}x$. Part (ii) can be obtained by letting $a_m = m$ and $b_m = \sin(m-1)x - \sin mx$ (or similarly $b_m = \cos\left(m - \frac{1}{2}\right)x$) in the stem which simplifies to give $p = -\frac{1}{4}n$ and $q = \frac{1}{4}(n+1)$.

$$\sum_{m=1}^{n} a_m(b_{m+1} - b_m) + \sum_{m=1}^{n} b_{m+1}(a_{m+1} - a_m)$$
$$= \sum_{m=1}^{n} (a_m b_{m+1} - a_m b_m + b_{m+1}a_{m+1} - b_{m+1}a_m)$$

M1

$$= \sum_{m=1}^{n} (-a_m b_m + b_{m+1}a_{m+1}) = a_{n+1}b_{n+1} - a_1 b_1$$

M1

Hence,

$$\sum_{m=1}^{n} a_m(b_{m+1} - b_m) = a_{n+1}b_{n+1} - a_1 b_1 - \sum_{m=1}^{n} b_{m+1}(a_{m+1} - a_m)$$

A1* (3)

(i) Let $a_m = 1$ (or any constant) and $b_m = \sin mx$, **M1**

then

$$\sum_{m=1}^{n} (\sin(m+1)x - \sin mx) = \sin(n+1)x - \sin x - \sum_{m=1}^{n} \sin(m+1)x \quad (1-1)$$

M1 A1

So

$$\sum_{m=1}^{n} 2\cos\left(m + \frac{1}{2}\right)x \, \sin\frac{1}{2}x = (\sin(n+1)x - \sin x)$$

M1 A1

and therefore

$$\sum_{m=1}^{n} \cos\left(m+\frac{1}{2}\right)x = \frac{1}{2}(\sin(n+1)x - \sin x)\csc\frac{1}{2}x$$

A1* (6)

(ii) Let $a_m = m$ and $b_m = \sin(m-1)x - \sin mx$, **M1**

then

$$b_{m+1} - b_m = (\sin mx - \sin(m+1)x) - (\sin(m-1)x - \sin mx)$$

$$= -2\cos\left(m+\frac{1}{2}\right)x\ \sin\frac{1}{2}x + \ 2\cos\left(m-\frac{1}{2}\right)x\ \sin\frac{1}{2}x$$

M1 A1

$= 4\sin mx\ \sin\frac{1}{2}x\sin\frac{1}{2}x$ **M1 A1**

Thus, using the stem

$$\sum_{m=1}^{n} m \times 4\sin mx\ \sin^2\frac{1}{2}x$$

$$= (n+1)(\sin nx - \sin(n+1)x) - 1\times(\sin(0\times x) - \sin x)$$

$$- \sum_{m=1}^{n}(\sin mx - \sin(m+1)x)$$

M1 A1

So

$$4\sin^2\frac{1}{2}x\sum_{m=1}^{n} m\sin mx\ = (n+1)(\sin nx - \sin(n+1)x) + \sin x - \sin x + \sin(n+1)x$$

M1 A1

$$4\sin^2\frac{1}{2}x\sum_{m=1}^{n} m\sin mx\ = (n+1)\sin nx - n\sin(n+1)x$$

Thus

$$\sum_{m=1}^{n} m \sin mx = (p \sin nx + q \sin(n+1)x) \csc^2 \frac{1}{2}x$$

where

$$p = -\frac{1}{4}n$$

A1

and

$$q = \frac{1}{4}(n+1)$$

A1 (11)

Alternatively, let $a_m = m$ and $b_m = \cos\left(m - \frac{1}{2}\right)x$, using stem, **M1**

$$\sum_{m=1}^{n} m\left(\cos\left(m+\frac{1}{2}\right)x - \cos\left(m-\frac{1}{2}\right)x\right)$$

$$= (n+1)\cos\left(n+\frac{1}{2}\right)x - \cos\frac{1}{2}x - \sum_{m=1}^{n} \cos\left(m+\frac{1}{2}\right)x$$

M1 A1

So,

$$\sum_{m=1}^{n} -2m \sin mx \ \sin\frac{1}{2}x$$

$$= (n+1)\cos\left(n+\frac{1}{2}\right)x - \cos\frac{1}{2}x - \frac{1}{2}(\sin(n+1)x - \sin x)\csc\frac{1}{2}x$$

M1 A1

$$= \csc\frac{1}{2}x\left((n+1)\cos\left(n+\frac{1}{2}\right)x \sin\frac{1}{2}x - \sin\frac{1}{2}x\cos\frac{1}{2}x - \frac{1}{2}(\sin(n+1)x - \sin x)\right)$$

M1 A1

$$= \frac{1}{2}\csc\frac{1}{2}x\left(2(n+1)\cos\left(n+\frac{1}{2}\right)x\sin\frac{1}{2}x - 2\sin\frac{1}{2}x\cos\frac{1}{2}x - (\sin(n+1)x - \sin x)\right)$$

$$= \frac{1}{2}\csc\frac{1}{2}x\left((n+1)(\sin(n+1)x - \sin nx) - \sin x - \sin(n+1)x + \sin x\right)$$

M1 A1

$$= \frac{1}{2}\csc\frac{1}{2}x\,(n\sin(n+1)x - (n+1)\sin nx)$$

giving result as before.