STEP III, 2017, Q8

8 Prove that, for any numbers $a_1, a_2, \ldots,$ and $b_1, b_2, \ldots,$ and for $n \geqslant 1$,

$$\sum_{m=1}^{n} a_m(b_{m+1} - b_m) = a_{n+1}b_{n+1} - a_1b_1 - \sum_{m=1}^{n} b_{m+1}(a_{m+1} - a_m)\,.$$

(i) By setting $b_m = \sin mx$, show that

$$\sum_{m=1}^{n} \cos(m + \tfrac{1}{2})x = \tfrac{1}{2}\big(\sin(n+1)x - \sin x\big)\operatorname{cosec}\tfrac{1}{2}x\,.$$

Note: $\sin A - \sin B = 2\cos\left(\dfrac{A+B}{2}\right)\sin\left(\dfrac{A-B}{2}\right).$

(ii) Show that

$$\sum_{m=1}^{n} m\sin mx = \big(p\sin(n+1)x + q\sin nx\big)\operatorname{cosec}^2\tfrac{1}{2}x\,,$$

where p and q are to be determined in terms of n.

Note: $2\sin A\sin B = \cos(A-B) - \cos(A+B)\,;$

$2\cos A\sin B = \sin(A+B) - \sin(A-B)\,.$