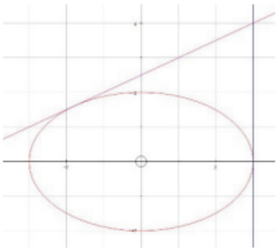
STEP III, 2017, Q7 MS

Question 7

Showing T lies on the ellipse is merely a matter of substituting T's coordinates into the left-hand side of the ellipse equation and simplifying to equal 1. The first result of (i) is tantamount to finding the equation of the tangent at T. Parametric differentiation leads to $\frac{\mathrm{dy}}{\mathrm{dx}} = -\frac{b(1-t^2)}{2at}$ giving L as $y - \frac{2bt}{(1+t^2)} = -\frac{b(1-t^2)}{2at}\left(x - \frac{a(1-t^2)}{(1+t^2)}\right)$ which (X, Y) satisfies, thus simplifying to the required result. The deduction can be made by requiring that the discriminant of the quadratic in t is positive and geometrically (X, Y) is a point outside the ellipse. Relaxing the restriction on the value of X, $X = \pm a$, so the inequality implies $Y \neq 0$ and thus there is a vertical tangent and another with one possible configuration as shown.



The first result of (ii) is obtained by considering p and q to be the roots of the quadratic in t, and hence being able to write down their product. Similarly, $p + q = \frac{2aY}{(a+X)b}$. The final result is obtained by finding expressions for y_1 and y_2 in terms of p and q respectively (without loss of generality), imposing the condition on y_1 and y_2 to get an equation in p and q, and then using the first two results from this part of the question to substitute for the product and sum of p and q.

$$\frac{\frac{a^2(1-t^2)^2}{(1+t^2)^2}}{a^2} + \frac{\frac{4b^2t^2}{(1+t^2)^2}}{b^2} = \frac{(1-t^2)^2+4t^2}{(1+t^2)^2} = \frac{1-2t^2+t^4+4t^2}{1+2t^2+t^4} = 1 \quad \textbf{B1 (1)}$$

(i) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow \frac{2x}{a^2}+\frac{2y\frac{dy}{dx}}{b^2}=0$ **M1**

$\frac{dy}{dx}=-\frac{b^2x}{a^2y}=-\frac{b^2a(1-t^2)(1+t^2)}{a^2(1+t^2)2bt}=-\frac{b(1-t^2)}{2at}$ **M1 A1**

So L is $y-\frac{2bt}{(1+t^2)}=-\frac{b(1-t^2)}{2at}\left(x-\frac{a(1-t^2)}{(1+t^2)}\right)$ **M1**

$2at(1+t^2)y-4abt^2=-bx(1-t^2)(1+t^2)+ab(1-t^2)^2$

$2at(1+t^2)y+bx(1-t^2)(1+t^2)=ab(1-t^2)^2+4abt^2=ab(1+t^2)^2$

Thus $2aty+bx(1-t^2)=ab(1+t^2)$ **M1**

and as (X,Y) lies on this line $2atY+bX(1-t^2)=ab(1+t^2)$

$0=(a+X)bt^2-2atY+b(a-X)$ **A1* (6)**

For there to be two distinct lines, there need to be two values of t.

So the discriminant must be positive, $(-2aY)^2-4(a+X)bb(a-X)>0$ **M1**

$4a^2Y^2>4b^2(a^2-X^2)$

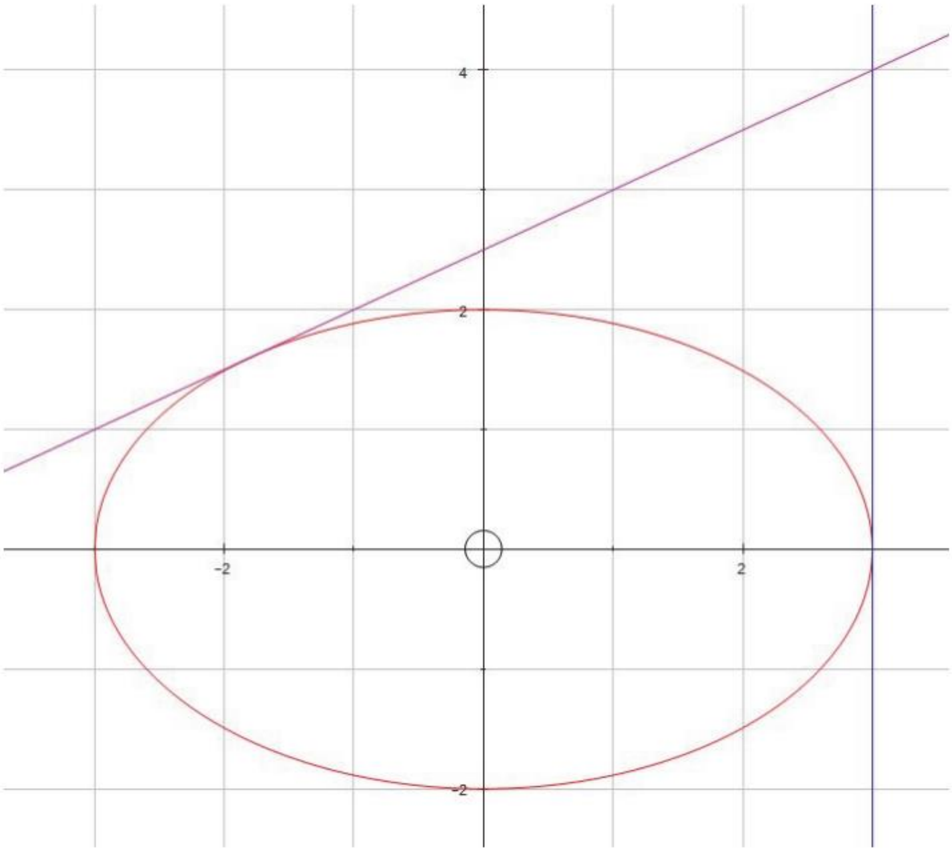
$a^2Y^2>(a^2-X^2)b^2$ **A1***

$\frac{Y^2}{b^2}>1-\frac{X^2}{a^2}$

$\frac{X^2}{a^2}+\frac{Y^2}{b^2}>1$ so (X,Y) lies outside the ellipse. **B1 (3)**

However, if $X^2=a^2$, $=\pm a$, one tangent is at $t=0$ or $t=\infty$, a vertical line. **E1**

If $\frac{X^2}{a^2}+\frac{Y^2}{b^2}>1$, then $Y\neq 0$. **E1**



G1 (3)

(ii) p and q are the roots of $0 = (a+X)bt^2 - 2atY + b(a-X)$

So $p+q = \frac{2aY}{(a+X)b}$ and $pq = \frac{b(a-X)}{(a+X)b}$ **M1**

Thus $(a+X)pq = a - X$ and $(a+X)(p+q)b = 2aY$ **A1 A1 (3)**

Without loss of generality $(0, y_1)$ lies on $(a+x)bp^2 - 2apy + b(a-x) = 0$

and $(0, y_2)$ lies on $(a+x)bq^2 - 2aqy + b(a-x) = 0$

So $abp^2 - 2apy_1 + ab = 0$, that is $bp^2 - 2py_1 + b = 0$ **M1**

and $bq^2 - 2qy_2 + b = 0$

As $y_1 + y_2 = 2b$, $\frac{bp^2+b}{2p} + \frac{bq^2+b}{2q} = 2b$ **M1**

$$\frac{p^2+1}{p} + \frac{q^2+1}{q} = 4$$

$$p + q + \frac{p+q}{pq} = 4$$

$$\frac{2aY}{(a+X)b} + \frac{\frac{2aY}{(a+X)b}}{\frac{a-X}{a+X}} = 4 \quad \textbf{M1}$$

$$\frac{2aY}{a+X} + \frac{2aY}{a-X} = 4b$$

$$2aY(a - X + a + X) = 4(a - X)(a + X)b$$

$$4a^2Y = 4(a^2 - X^2)b$$

$$\frac{Y}{b} = 1 - \frac{X^2}{a^2}$$

$$\frac{X^2}{a^2} + \frac{Y}{b} = 1 \quad \textbf{A1* (4)}$$