STEP III, 2017, Q7

7 Show that the point T with coordinates

$$\left(\frac{a(1-t^2)}{1+t^2}\,,\ \frac{2bt}{1+t^2}\right) \qquad (*)$$

(where a and b are non-zero) lies on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\,.$$

(i) The line L is the tangent to the ellipse at T. The point (X,Y) lies on L, and $X^2 \neq a^2$. Show that

$$(a+X)bt^2 - 2aYt + b(a-X) = 0\,.$$

Deduce that if $a^2Y^2 > (a^2-X^2)b^2$, then there are two distinct lines through (X,Y) that are tangents to the ellipse. Interpret this result geometrically. Show, by means of a sketch, that the result holds also if $X^2 = a^2$.

(ii) The distinct points P and Q are given by $(*)$, with $t=p$ and $t=q$, respectively. The tangents to the ellipse at P and Q meet at the point with coordinates (X,Y), where $X^2 \neq a^2$. Show that

$$(a+X)pq = a-X$$

and find an expression for $p+q$ in terms of a, b, X and Y.

Given that the tangents meet the y-axis at points $(0,y_1)$ and $(0,y_2)$, where $y_1+y_2=2b$, show that

$$\frac{X^2}{a^2}+\frac{Y}{b}=1\,.$$