STEP III, 2017, Q6 MS

Question 6

The appropriate substitution for part (i) was $u = v^{-1}$, and having changed variable the resulting integral has limits x^{-1} and ∞ , which can be expressed as the difference of two integrals with these as their upper limits and zero as their lower. To obtain $\frac{\mathrm{d}v}{\mathrm{d}u}$ in part (ii), one method is to make a (the constant) the subject of the formula and to differentiate with respect to u; an alternative is to differentiate v directly, to multiply numerator and denominator by $(1 + u^2)$, to expand the numerator and then to express it as $(1 - au)^2 + (u + a)^2$ leading to the desired result. Applying this substitution to the defined $\mathrm{T}(x)$, results in an integral which can be expressed again as the difference of two integrals as in part (i). Taking the result of (i) and rearranging to make $\mathrm{T}(x^{-1})$ the subject, $\mathrm{T}(x)$ can be substituted for using the result just found and in turn $\mathrm{T}(a)$ can be replaced using the result of (i) with x as a. The final result of (ii) is achieved by letting $y = x^{-1}$, and $b = a^{-1}$ in that just found. Throughout, it is important that the conditions expressed as inequalities are substantiated. To find $\mathrm{T}(\sqrt{3})$ in (iii), apply the final result of (ii) letting $y = b = \sqrt{3}$, whereas to find $\mathrm{T}(\sqrt{2} - 1)$, let $x = \sqrt{2} - 1$ and $a = 1$ in the result

$\mathrm{T}(x) = \mathrm{T}\left(\frac{x+a}{1-ax}\right) - \mathrm{T}(a)$, deal with $\mathrm{T}(\sqrt{2} + 1)$ by letting $x = \sqrt{2} + 1$ in the result of part (i) and then $\mathrm{T}(1)$ using the same result but with $x = 1$.

(i) $T(x) = \int_0^x \frac{1}{1+u^2} du$

Let $u = v^{-1}$, $\frac{du}{dv} = -v^{-2}$

B1

So

$$T(x) = \int_{\infty}^{x^{-1}} \frac{1}{1+v^{-2}} \times -v^{-2} dv = \int_{x^{-1}}^{\infty} \frac{1}{v^2+1} dv = \int_0^{\infty} \frac{1}{1+u^2} du - \int_0^{x^{-1}} \frac{1}{1+u^2} du$$

M1 **M1**

$T(x) = T(\infty) - T(x^{-1})$ **A1* (4)**

(ii) $v = \frac{u+a}{1-au} \Leftrightarrow v - auv = u + a \Leftrightarrow v - u = a(1 + uv) \Leftrightarrow a = \frac{v-u}{1+uv}$

M1

$0 = \frac{(1+uv)\left(\frac{dv}{du}-1\right)-(v-u)\left(u\frac{dv}{du}+v\right)}{(1+uv)^2}$ **M1**

$\frac{dv}{du}(1 + uv - uv + u^2) = 1 + uv + v^2 - uv$

$\frac{dv}{du} = \frac{1+v^2}{1+u^2}$ **A1* (3)**

Alternatively,

$$v = \frac{u+a}{1-au} \Leftrightarrow \frac{dv}{du} = \frac{(1-au)+a(u+a)}{(1-au)^2} = \frac{1+a^2}{(1-au)^2} = \frac{(1+a^2)(1+u^2)}{(1-au)^2(1+u^2)}$$

M1

$$= \frac{(1-au)^2+(u+a)^2}{(1-au)^2(1+u^2)} = \frac{1+v^2}{1+u^2}$$

M1 **A1**

$T(x) = \int_0^x \frac{1}{1+u^2}du = \int_a^{\frac{x+a}{1-ax}} \frac{1}{1+u^2}\frac{1+u^2}{1+v^2}dv = \int_a^{\frac{x+a}{1-ax}} \frac{1}{1+v^2}dv = \int_0^{\frac{x+a}{1-ax}} \frac{1}{1+v^2}dv - \int_0^a \frac{1}{1+v^2}dv$

M1 **M1**

$T(x) = T\left(\frac{x+a}{1-ax}\right) - T(a)$ **A1* (3)**

As $T(x) = T(\infty) - T(x^{-1})$, $T(a) = T(\infty) - T(a^{-1})$

$$T(x^{-1}) = T(\infty) - T(x) = T(\infty) - \left(T\left(\frac{x+a}{1-ax}\right) - T(a)\right) = T(\infty) - \left(T\left(\frac{x+a}{1-ax}\right) - \left(T(\infty) - T(a^{-1})\right)\right)$$

M1 **M1**

Thus

$T(x^{-1}) = 2T(\infty) - T\left(\frac{x+a}{1-ax}\right) - T(a^{-1})$ **A1* (3)**

Let $y = x^{-1}$, $= a^{-1}$, then $x < \frac{1}{a}$ implies $\frac{1}{y} < b$ which is $y > \frac{1}{b}$ **M1**

$T(y) = 2T(\infty) - T\left(\frac{y^{-1}+b^{-1}}{1-b^{-1}y^{-1}}\right) - T(b) = 2T(\infty) - T\left(\frac{b+y}{by-1}\right) - T(b)$ **A1* (2)**

(iii) Using $T(y) = 2T(\infty) - T\left(\frac{b+y}{by-1}\right) - T(b)$ with $y = b = \sqrt{3}$ **M1**

$T(\sqrt{3}) = 2T(\infty) - T\left(\frac{\sqrt{3}+\sqrt{3}}{\sqrt{3}\sqrt{3}-1}\right) - T(\sqrt{3})$

$T(\sqrt{3}) = 2T(\infty) - T(\sqrt{3}) - T(\sqrt{3})$

$3T(\sqrt{3}) = 2T(\infty) \Leftrightarrow T(\sqrt{3}) = \frac{2}{3}T(\infty)$ **A1* (2)**

Using $T(x) = T(\infty) - T(x^{-1})$ with $x = 1$,

$T(1) = T(\infty) - T(1)$ and so $T(1) = \frac{1}{2}T(\infty)$ **B1**

Using $T(x) = T\left(\frac{x+a}{1-ax}\right) - T(a)$ with $x = \sqrt{2} - 1$ and $a = 1$ **M1**

$T(\sqrt{2} - 1) = T\left(\frac{\sqrt{2}-1+1}{1-(\sqrt{2}-1)}\right) - T(1)$

$T(\sqrt{2} - 1) = T\left(\frac{\sqrt{2}}{2-\sqrt{2}}\right) - T(1) = T\left(\frac{1}{\sqrt{2}-1}\right) - T(1) = T(\sqrt{2}+1) - T(1)$

Using $T(x) = T(\infty) - T(x^{-1})$, $T(\sqrt{2}+1) = T(\infty) - T(\sqrt{2}-1)$

So $T(\sqrt{2} - 1) = T(\infty) - T(\sqrt{2}-1) - T(1)$

$2T(\sqrt{2} - 1) = T(\infty) - T(1) = T(\infty) - \frac{1}{2}T(\infty)$

$T(\sqrt{2} - 1) = \frac{1}{4}T(\infty)$ **A1* (3)**

Alternatively, using $T(x) = T\left(\frac{x+a}{1-ax}\right) - T(a)$ with $x = a = \sqrt{2} - 1$

$$T(\sqrt{2} - 1) = T\left(\frac{2(\sqrt{2}-1)}{1-(\sqrt{2}-1)^2}\right) - T(\sqrt{2}-1) = T\left(\frac{2(\sqrt{2}-1)}{2(\sqrt{2}-1)}\right) - T(\sqrt{2}-1)$$

Therefore $2T(\sqrt{2} - 1) = T(1)$ and so $T(\sqrt{2} - 1) = \frac{1}{2}T(1) = \frac{1}{4}T(\infty)$