# STEP III, 2017, Q6

**6** *In this question, you are not permitted to use any properties of trigonometric functions or inverse trigonometric functions.*

The function T is defined for $x > 0$ by

$$\mathrm{T}(x) = \int_0^x \frac{1}{1+u^2}\,\mathrm{d}u\,,$$

and $T_\infty = \displaystyle\int_0^\infty \frac{1}{1+u^2}\,\mathrm{d}u$ (which has a finite value).

**(i)** By making an appropriate substitution in the integral for $\mathrm{T}(x)$, show that

$$\mathrm{T}(x) = \mathrm{T}_\infty - \mathrm{T}(x^{-1})\,.$$

**(ii)** Let $v = \dfrac{u+a}{1-au}$, where $a$ is a constant. Verify that, for $u \neq a^{-1}$,

$$\frac{\mathrm{d}v}{\mathrm{d}u} = \frac{1+v^2}{1+u^2}\,.$$

Hence show that, for $a > 0$ and $x < \dfrac{1}{a}$,

$$\mathrm{T}(x) = \mathrm{T}\left(\frac{x+a}{1-ax}\right) - \mathrm{T}(a)\,.$$

Deduce that

$$\mathrm{T}(x^{-1}) = 2\mathrm{T}_\infty - \mathrm{T}\left(\frac{x+a}{1-ax}\right) - \mathrm{T}(a^{-1})$$

and hence that, for $b > 0$ and $y > \dfrac{1}{b}$,

$$\mathrm{T}(y) = 2\mathrm{T}_\infty - \mathrm{T}\left(\frac{y+b}{by-1}\right) - \mathrm{T}(b)\,.$$

**(iii)** Use the above results to show that $\mathrm{T}(\sqrt{3}) = \frac{2}{3}\mathrm{T}_\infty$ and $\mathrm{T}(\sqrt{2}-1) = \frac{1}{4}\mathrm{T}_\infty$.