

## STEP III, 2017, Q6

- 6 *In this question, you are not permitted to use any properties of trigonometric functions or inverse trigonometric functions.*

The function  $T$  is defined for  $x > 0$  by

$$T(x) = \int_0^x \frac{1}{1+u^2} du,$$

and  $T_\infty = \int_0^\infty \frac{1}{1+u^2} du$  (which has a finite value).

- (i) By making an appropriate substitution in the integral for  $T(x)$ , show that

$$T(x) = T_\infty - T(x^{-1}).$$

- (ii) Let  $v = \frac{u+a}{1-au}$ , where  $a$  is a constant. Verify that, for  $u \neq a^{-1}$ ,

$$\frac{dv}{du} = \frac{1+v^2}{1+u^2}.$$

Hence show that, for  $a > 0$  and  $x < \frac{1}{a}$ ,

$$T(x) = T\left(\frac{x+a}{1-ax}\right) - T(a).$$

Deduce that

$$T(x^{-1}) = 2T_\infty - T\left(\frac{x+a}{1-ax}\right) - T(a^{-1})$$

and hence that, for  $b > 0$  and  $y > \frac{1}{b}$ ,

$$T(y) = 2T_\infty - T\left(\frac{y+b}{by-1}\right) - T(b).$$

- (iii) Use the above results to show that  $T(\sqrt{3}) = \frac{2}{3}T_\infty$  and  $T(\sqrt{2}-1) = \frac{1}{4}T_\infty$ .



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