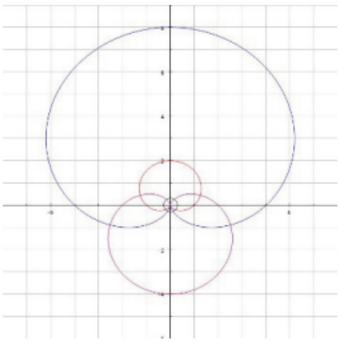
STEP III, 2017, Q5 MS

Question 5

Converting polar coordinates to Cartesian and differentiating each of x and y with respect to θ gives a fraction which simplifies to $\frac{dy}{dx} = \frac{f+f' \tan\theta}{-f\tan\theta+f'}$ once the numerator and denominator have been divided by $\cos\theta$. Setting the product of two such expressions for f and g equal to negative 1 and simplifying leads to $(fg + f'g')\sec^2\theta = 0$ and hence the desired result. Substituting for g in the displayed result and solving the resulting first order differential equation for f (by integrating factor or separation of variables) leads to $f(\theta) = \left(\frac{k\cos^2\theta}{1+\sin\theta}\right)$ which simplifies by eliminating $\cos^2\theta$ in favour of $\sin^2\theta$, and hence using the given point, $f(\theta) = 2(1-\sin\theta)$.



$y = r\sin\theta$

$\frac{dy}{d\theta} = r\cos\theta + \frac{dr}{d\theta}\sin\theta = f\cos\theta + f'\sin\theta$ **M1**

$x = r\cos\theta$

$\frac{dx}{d\theta} = -r\sin\theta + \frac{dr}{d\theta}\cos\theta = -f\sin\theta + f'\cos\theta$ **M1**

$\frac{dy}{dx} = \frac{f\cos\theta + f'\sin\theta}{-f\sin\theta + f'\cos\theta} = \frac{f+f'\tan\theta}{-f\tan\theta+f'}$ **M1 A1 (4)**

$\frac{f+f'\tan\theta}{-f\tan\theta+f'} \times \frac{g+g'\tan\theta}{-g\tan\theta+g'} = -1$ **M1**

$fg + f'g\tan\theta + fg'\tan\theta + f'g'\tan^2\theta = -fg\tan^2\theta + f'g\tan\theta + fg'\tan\theta - f'g'$

$(fg + f'g')\sec^2\theta = 0$ **M1**

$fg + f'g' = 0$ **A1* (3)**

$g(\theta) = a(1+\sin\theta)$

$g'(\theta) = a\cos\theta$

So $f'a\cos\theta + fa(1+\sin\theta) = 0$ **M1**

$\frac{f'}{f} = -\frac{(1+\sin\theta)}{\cos\theta} = -\sec\theta - \tan\theta$ **A1**

$\ln f = -\ln(\sec\theta + \tan\theta) + \ln\cos\theta + c = \ln\left(\frac{k\cos\theta}{\sec\theta+\tan\theta}\right) = \ln\left(\frac{k\cos^2\theta}{1+\sin\theta}\right)$ **M1 A1**

$f(\theta) = \left(\frac{k\cos^2\theta}{1+\sin\theta}\right) = \frac{k(1-\sin^2\theta)}{1+\sin\theta} = k(1-\sin\theta)$ **M1 A1)**

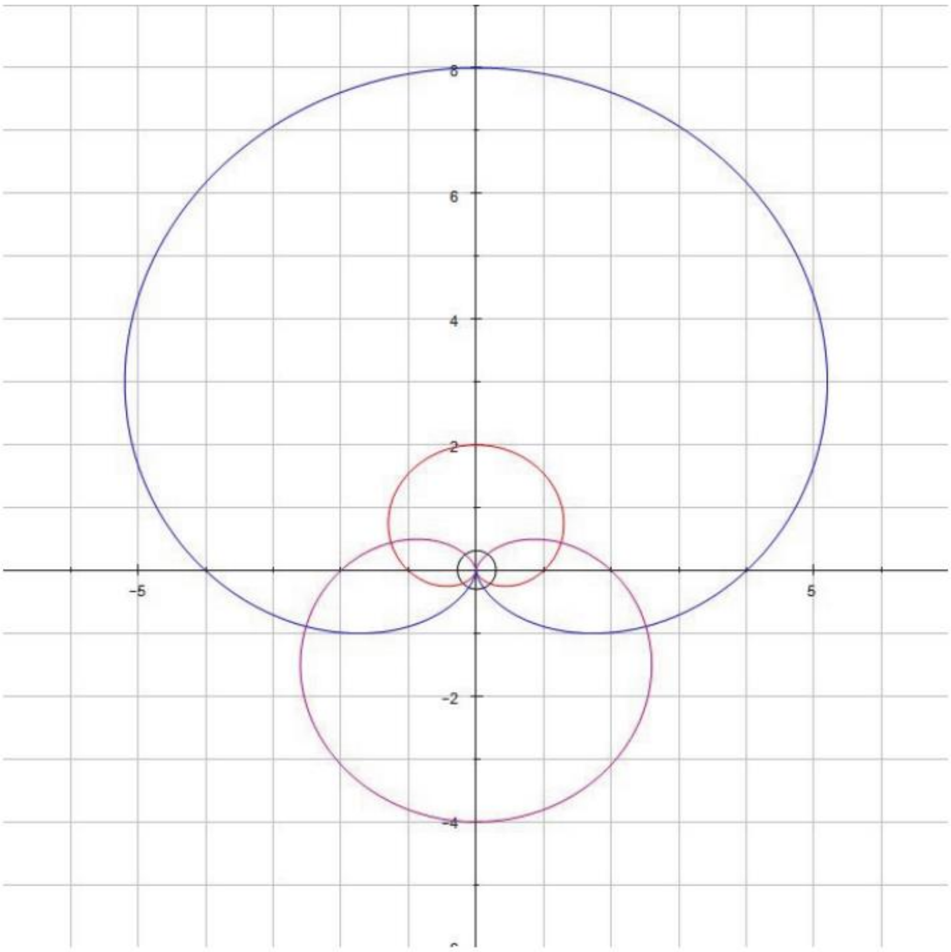
Alternatively, $\frac{f'}{f} = -\frac{(1+\sin\theta)}{\cos\theta} = -\frac{\cos\theta}{(1-\sin\theta)}$ **M1 A1**

$\ln f = \ln((1-\sin\theta)) + c = \ln(k(1-\sin\theta))$ **M1**

and hence $f(\theta) = k(1-\sin\theta)$ **A1**

$r = 4, \theta = -\frac{1}{2}\pi$ so $4 = 2k$ **M1**

Thus $f(\theta) = 2(1-\sin\theta)$ **A1 (8)**



G1 G1 dG1 G1 G1(5)