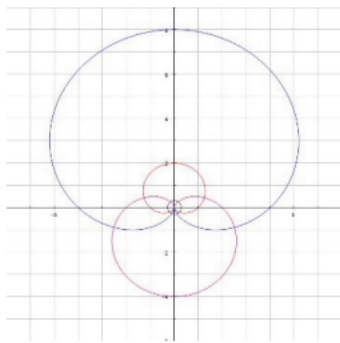


STEP III, 2017, Q5 MS

Question 5

Converting polar coordinates to Cartesian and differentiating each of x and y with respect to θ gives a fraction which simplifies to $\frac{dy}{dx} = \frac{f+f' \tan \theta}{-f \tan \theta + f'}$ once the numerator and denominator have been divided by $\cos \theta$. Setting the product of two such expressions for f and g equal to negative 1 and simplifying leads to $(fg + f'g') \sec^2 \theta = 0$ and hence the desired result. Substituting for g in the displayed result and solving the resulting first order differential equation for f (by integrating factor or separation of variables) leads to $f(\theta) = \left(\frac{k \cos^2 \theta}{1 + \sin \theta}\right)$ which simplifies by eliminating $\cos^2 \theta$ in favour of $\sin^2 \theta$, and hence using the given point, $f(\theta) = 2(1 - \sin \theta)$.



$$y = r \sin \theta$$

$$\frac{dy}{d\theta} = r \cos \theta + \frac{dr}{d\theta} \sin \theta = f \cos \theta + f' \sin \theta \quad \mathbf{M1}$$

$$x = r \cos \theta$$

$$\frac{dx}{d\theta} = -r \sin \theta + \frac{dr}{d\theta} \cos \theta = -f \sin \theta + f' \cos \theta \quad \mathbf{M1}$$

$$\frac{dy}{dx} = \frac{f \cos \theta + f' \sin \theta}{-f \sin \theta + f' \cos \theta} = \frac{f + f' \tan \theta}{-f \tan \theta + f'} \quad \mathbf{M1 \ A1 \ (4)}$$



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$$\frac{f+f' \tan \theta}{-f \tan \theta+f'} \times \frac{g+g' \tan \theta}{-g \tan \theta+g'} = -1 \quad \text{M1}$$

$$fg + f'g \tan \theta + fg' \tan \theta + f'g' \tan^2 \theta = -fg \tan^2 \theta + f'g \tan \theta + fg' \tan \theta - f'g'$$

$$(fg + f'g') \sec^2 \theta = 0 \quad \text{M1}$$

$$fg + f'g' = 0 \quad \text{A1* (3)}$$

$$g(\theta) = a(1 + \sin \theta)$$

$$g'(\theta) = a \cos \theta$$

$$\text{So } f'a \cos \theta + fa(1 + \sin \theta) = 0 \quad \text{M1}$$

$$\frac{f'}{f} = -\frac{(1+\sin \theta)}{\cos \theta} = -\sec \theta - \tan \theta \quad \text{A1}$$

$$\ln f = -\ln(\sec \theta + \tan \theta) + \ln \cos \theta + c = \ln\left(\frac{k \cos \theta}{\sec \theta + \tan \theta}\right) = \ln\left(\frac{k \cos^2 \theta}{1 + \sin \theta}\right) \quad \text{M1 A1}$$

$$f(\theta) = \left(\frac{k \cos^2 \theta}{1 + \sin \theta}\right) = \frac{k(1 - \sin^2 \theta)}{1 + \sin \theta} = k(1 - \sin \theta) \quad \text{M1 A1}$$

$$\text{Alternatively, } \frac{f'}{f} = -\frac{(1+\sin \theta)}{\cos \theta} = -\frac{\cos \theta}{(1-\sin \theta)} \quad \text{M1 A1}$$

$$\ln f = \ln((1 - \sin \theta)) + c = \ln(k(1 - \sin \theta)) \quad \text{M1}$$

$$\text{and hence } f(\theta) = k(1 - \sin \theta) \quad \text{A1}$$

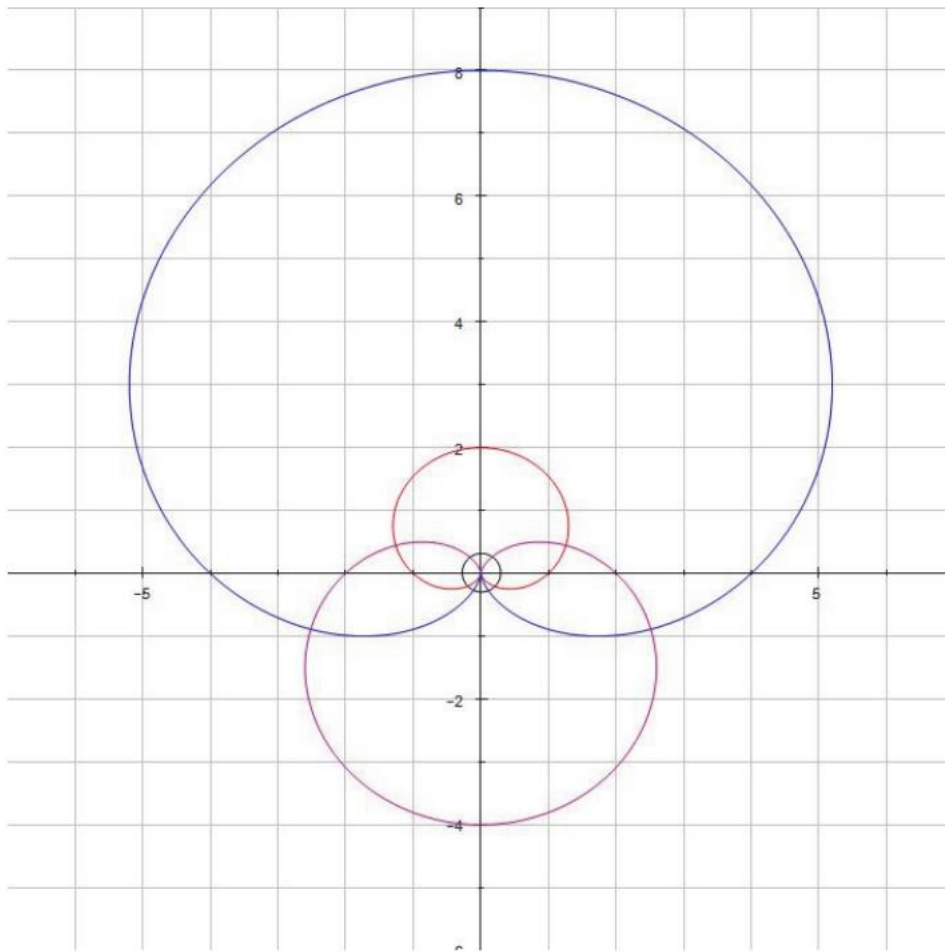
$$r = 4, \theta = -\frac{1}{2}\pi \text{ so } 4 = 2k \quad \text{M1}$$

$$\text{Thus } f(\theta) = 2(1 - \sin \theta) \quad \text{A1 (8)}$$



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G1 G1 dG1 G1 G1(5)



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