STEP III, 2017, Q4 MS

Question 4

Letting $\log_a f(x) = z$, $f(x) = a^z = e^{z \ln a}$ (using the result from the formula book) and so, $\ln f(x) = z \ln a = \ln a \ \log_a f(x)$, which substituted in the geometric mean definition simplifies rapidly to the required result. Part (ii) can be obtained by substituting for $h(x)$ in the expression for $H(y)$ and then manipulating using the logarithm of a product. Part (iii) can be obtained using the result of part (i) with $a = b$, and then checking separately that the simple case $b = 1$ works. In part (iv), setting the defined expression for the geometric mean of $f(x)$ equal to $\sqrt{f(y)}$, and taking the logarithm of both expressions yields, after minor rearrangement, $\int_0^y \ln f(x) dx = \frac{y}{2} \ln f(y)$. Differentiating this with respect to y, and again rearranging, leads to the first order separable differential equation $\frac{f'(y)}{f(y) \ln f(y)} = \frac{1}{y}$, which integrates to $\ln \ln f(y) = \ln y + c$ leading to the desired result by judicious choice of c.

(i) $e^{x \ln a} = a^x$ (formula book)

So if $\log_a f(x) = z$

$f(x) = a^z = e^{z \ln a}$ **E1**

and so $\ln f(x) = z \ln a = \ln a \ \log_a f(x)$ **B1**

Therefore,

$$e^{\frac{1}{y} \int_0^y \ln f(x) dx} = e^{\frac{1}{y} \int_0^y \ln a \ \log_a f(x) \, dx} = e^{\frac{1}{y} \ln a \int_0^y \log_a f(x) \, dx}$$

M1 **M1**

Thus, $F(y) = a^{\frac{1}{y} \int_0^y \log_a f(x) \, dx}$ **A1* (5)**

(ii) $H(y)=e^{\frac{1}{y}\int_0^y \ln h(x)dx}=e^{\frac{1}{y}\int_0^y \ln(f(x)g(x))dx}$ **M1**

$=e^{\frac{1}{y}\int_0^y \ln f(x)+\ln g(x)dx}$

$=e^{\frac{1}{y}\left(\int_0^y \ln f(x)dx+\int_0^y \ln g(x)dx\right)}$ **M1**

$=\ e^{\frac{1}{y}\int_0^y \ln f(x)dx}\ e^{\frac{1}{y}\int_0^y \ln g(x)dx}=F(y)G(y)$

M1 **A1* (4)**

(iii) Let $f(x)=b^x$,

Then $F(y)=e^{\frac{1}{y}\int_0^y \ln b^x dx}=e^{\frac{1}{y}\int_0^y x\ln b\ dx}=e^{\frac{1}{y}\ln b}\ \int_0^y xdx$

M1 **M1**

$=e^{\frac{1}{y}\ln b}\left[\frac{1}{2}x^2\right]_0^y=e^{\frac{1}{y}\ln b}\ \frac{1}{2}y^2=e^{\frac{1}{2}y\ln b}=b^{\frac{1}{2}y}=\sqrt{b^y}$

A1 **M1** **A1* (5)**

(iv) $e^{\frac{1}{y}\int_0^y \ln f(x)dx}=\sqrt{f(y)}$

$\frac{1}{y}\int_0^y \ln f(x)dx=\ln\sqrt{f(y)}=\frac{1}{2}\ln f(y)$

$\int_0^y \ln f(x)dx=\frac{y}{2}\ln f(y)$ **M1**

$\ln f(y)=\frac{1}{2}\ln f(y)+\frac{yf'(y)}{2f(y)}$ **M1**

$\frac{yf'(y)}{f(y)}=\ln f(y)$ so $\frac{f'(y)}{f(y)\ln f(y)}=\frac{1}{y}$ **M1**

Integrating $\ln\ln f(y)=\ln y+c=\ln y+\ln k=\ln ky$ **M1 A1**

$\ln f(y)=ky$

$f(y)=e^{ky}=e^{y\ln b}=b^y$

$f(x)=b^x$ **A1* (6)**