STEP III, 2017, Q4

4 For any function f satisfying $f(x) > 0$, we define the *geometric mean*, F, by

$$F(y) = e^{\frac{1}{y}\int_0^y \ln f(x)\,dx} \qquad (y > 0)\,.$$

(i) The function f satisfies $f(x) > 0$ and a is a positive number with $a \neq 1$. Prove that

$$F(y) = a^{\frac{1}{y}\int_0^y \log_a f(x)\,dx}\,.$$

(ii) The functions f and g satisfy $f(x) > 0$ and $g(x) > 0$, and the function h is defined by $h(x) = f(x)g(x)$. Their geometric means are F, G and H, respectively. Show that $H(y) = F(y)G(y)$.

(iii) Prove that, for any positive number b, the geometric mean of b^x is $\sqrt{b^y}$.

(iv) Prove that, if $f(x) > 0$ and the geometric mean of $f(x)$ is $\sqrt{f(y)}$, then $f(x) = b^x$ for some positive number b.