STEP III, 2017, Q3 MS

Question 3

Writing down the sum of the three roots of the cubic gives an expression which must be q in the quartic and $-A$ in the cubic. In the specific case, the cubic equation is

$y^3 - 3y^2 - 40y + 84 = 0$ which has roots -6, 2 and 7, and thus $\alpha\beta + \gamma\delta = 7$. Expanding the expression whose value is required in (ii) gives q without the two terms whose sum has just been found, and hence -4. Equally the product of those two terms is s (10), and so a quadratic equation for them gives $\alpha\beta = 5$, the larger of the two roots. Using the first result of (ii) with the knowledge that $(\alpha + \beta) + (\gamma + \delta) = p = 0$, and that

$\alpha\beta(\gamma + \delta) + \gamma\delta(\alpha + \beta) = 6$, leads to the results that α and β are the roots of $t^2 + 2t + 5 = 0$ and γ and δ are the roots of $t^2 - 2t + 2 = 0$. Hence the four roots of the quartic are $1 \pm i$, $-1 \pm 2i$.

$\alpha\beta + \gamma\delta + \alpha\gamma + \beta\delta + \alpha\delta + \beta\gamma = -A$ **M1**

$A = -q$ **A1 (2)**

(i) $y^3 - 3y^2 - 40y + 84 = 0$ **M1 A1**

$(y - 2)(y^2 - y - 42) = 0$ **M1**

$(y - 2)(y - 7)(y + 6) = 0$ **M1 A1**

So $\alpha\beta + \gamma\delta = 7$ **A1 (6)**

(ii) $(\alpha + \beta)(\gamma + \delta) = \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta = 3 - \alpha\beta - \gamma\delta = -4$

M1 **M1** **A1 (3)**

$(\alpha + \beta) + (\gamma + \delta) = 0$ **M1**

Thus $(\alpha + \beta)$ is a root of $t^2 - 4 = 0$ **M1**

So $\alpha + \beta = \pm 2$ **A1**

$$\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta = 6$$

$$\alpha\beta(\gamma + \delta) + \gamma\delta(\alpha + \beta) = 6$$

$2(\alpha\beta - \gamma\delta) = \pm 6$ **M1**

$\alpha\beta - \gamma\delta = 3$ as $\alpha\beta > \gamma\delta$ (and so $\alpha + \beta = -2$)

So $\alpha\beta = 5$ **A1 (5)**

Alternatively, $\alpha\beta\gamma\delta = 10$, **M1 A1** so $\alpha\beta$ and $\gamma\delta$ are the roots of

$t^2 - 7t + 10 = 0$ **M1 A1** and as $\alpha\beta > \gamma\delta$, $\alpha\beta = 5$ (and $\gamma\delta = 2$). **A1 (5)**

(iii) Thus α and β are the roots of $t^2 + 2t + 5 = 0$ and γ and δ are the roots of $t^2 - 2t + 2 = 0$ **M1 A1**

So $x = 1 \pm i$, $-1 \pm 2i$ **A1 A1 (4)**