STEP III, 2017, Q2 MS

Question 2

(i) is simply obtained by applying $e^{i\theta}$ to $z - a$. Using the result of (i) twice for SR and for a rotation about c and equating, both sides can be multiplied by $-e^{\frac{-i(\varphi+\theta)}{2}}$. In the case $\varphi + \theta = 2n\pi$, $\left(1 - e^{i(\varphi+\theta)}\right) = 0$, so c cannot be found, and then SR is a translation by $(b-a)\left(1 - e^{i\varphi}\right)$. If $RS = SR$, and if $\varphi + \theta = 2n\pi$, then

$(b-a)\left(1 - e^{i\varphi}\right) = (a-b)\left(1 - e^{i\theta}\right)$ and so either $a = b$ or if $a \neq b$, $\theta = 2m\pi$. If $\varphi + \theta \neq 2n\pi$, $a = b$, $\theta = 2n\pi$, or $\varphi = 2n\pi$

(i) $z' - a = e^{i\theta}(z - a)$ **M1**

Thus $z' = a + e^{i\theta}z - e^{i\theta}a = e^{i\theta}z + a\left(1 - e^{i\theta}\right)$ **A1* (2)**

(ii) $z'' = e^{i\varphi}z' + b\left(1 - e^{i\varphi}\right)$

$= e^{i\varphi}\left(e^{i\theta}z + a\left(1 - e^{i\theta}\right)\right) + b\left(1 - e^{i\varphi}\right)$ **M1 A1**

So $z'' = e^{i(\varphi+\theta)}z + \left(ae^{i\varphi} - ae^{i(\varphi+\theta)} + b - be^{i\varphi}\right)$

This is a rotation about c if

$c\left(1 - e^{i(\varphi+\theta)}\right) = ae^{i\varphi} - ae^{i(\varphi+\theta)} + b - be^{i\varphi}$ **M1**

If $\varphi + \theta = 2n\pi$, $\left(1 - e^{i(\varphi+\theta)}\right) = 0$, so c cannot be found. **E1**

Otherwise, multiplying by $-e^{\frac{-i(\varphi+\theta)}{2}}$,

$$c\left(e^{\frac{i(\varphi+\theta)}{2}}-e^{\frac{-i(\varphi+\theta)}{2}}\right)=a\left(e^{\frac{i(\varphi+\theta)}{2}}-e^{\frac{i(\varphi-\theta)}{2}}\right)+b\left(e^{\frac{i(\varphi-\theta)}{2}}-e^{\frac{-i(\varphi+\theta)}{2}}\right)$$

$2ci\sin\frac{1}{2}(\varphi+\theta)=2aie^{i\varphi/2}\sin\frac{1}{2}\theta+2bie^{-i\theta/2}\sin\frac{1}{2}\varphi$ **M1**

$c\sin\frac{1}{2}(\varphi+\theta)=ae^{i\varphi/2}\sin\frac{1}{2}\theta+be^{-i\theta/2}\sin\frac{1}{2}\varphi$ **A1* (6)**

If $\varphi+\theta=2n\pi$, $z''=z+\left(ae^{i\varphi}-a+b-be^{i\varphi}\right)$ **M1**

So $z''=z+(b-a)\left(1-e^{i\varphi}\right)$ **A1**

This is a translation by $(b-a)\left(1-e^{i\varphi}\right)$ **A1 (3)**

(iii) If RS = SR, and if $\varphi+\theta=2n\pi$, then

$$(b-a)\left(1-e^{i\varphi}\right)=(a-b)\left(1-e^{i\theta}\right)$$

M1

$$(a-b)\left(e^{i\theta}+e^{i\varphi}-2\right)=0$$

So $a=b$, or if $a\neq b$, $e^{i\theta}+e^{i(2n\pi-\theta)}-2=0$

A1

$2\cos\theta-2=0$ **M1**

Thus $\theta=2n\pi$ **A1 (4)**

If $\varphi+\theta\neq 2n\pi$

$ae^{i\varphi/2}\sin\frac{1}{2}\theta+be^{-i\theta/2}\sin\frac{1}{2}\varphi=be^{i\theta/2}\sin\frac{1}{2}\varphi+ae^{-i\varphi/2}\sin\frac{1}{2}\theta$ **M1**

$2i(a-b)\sin\frac{1}{2}\varphi\sin\frac{1}{2}\theta=0$ **A1**

So $a=b$, $\theta=2n\pi$, or $\varphi=2n\pi$

A1 **A1** **A1 (5)**