STEP III, 2017, Q1 MS

Question 1

The first result is simply obtained by expanding the bracketed expression on the right-hand side using the definition of the binomial coefficients, and then combining the fractions using the lowest common denominator. $\sum_{n=1}^{\infty} \frac{1}{{}^{n+r}C_{r+1}}$ is determined by employing the first result of the question, finding that the terms telescope and then observing that ${}^{n+r}C_r \to \infty$ as $n \to \infty$, to give the answer $\frac{r+1}{r}$. The deduced result is obtained by letting $r = 2$ in the previous result and subtracting the first term of the sum in the general result. The first inequality of (ii) can be obtained by expanding ${}^{n+1}C_3$ as $\frac{n^3-n}{3!}$, observing that $\frac{n^3-n}{3!} < \frac{n^3}{3!}$ and rearranging. Similarly, $\frac{20}{{}^{n+1}C_3} - \frac{1}{{}^{n+2}C_5} - \frac{5!}{n^3}$ is $\frac{-480}{n^3(n^2-1)(n^2-4)}$ which is negative as $n \geq 3$ and so the denominator is positive, leading to the second inequality. The first numerical inequality in the final result is obtained from the second inequality of part (ii) using the final result of (i) for the first summed term, the penultimate result of (i) for the second summed term (adjusting the index over which it is summed), and including the terms for $n = 1$ and $n = 2$. The second numerical inequality in the final line is obtained from the first inequality of part (ii), again including the two extra terms and using the final result of (i).

(i) $\frac{1}{{}^{n+r-1}C_r} - \frac{1}{{}^{n+r}C_r} = \frac{(n-1)!\,r!}{(n+r-1)!} - \frac{n!r!}{(n+r)!}$ **M1**

$= \frac{(n-1)!\,r!\,[(n+r)-n]}{(n+r)!}$

$= \frac{(n-1)!\,r!\,r}{(n+r)!}$ **M1**

$\therefore \frac{r+1}{r}\left(\frac{1}{{}^{n+r-1}C_r} - \frac{1}{{}^{n+r}C_r}\right) = \frac{r+1}{r}\frac{(n-1)!\,r!\,r}{(n+r)!}$

$= \frac{(n-1)!\,(r+1)!}{(n+r)!} = \frac{1}{{}^{n+r}C_{r+1}}$ **A1*** **(3)**

$\sum_{n=1}^{\infty} \frac{1}{{}^{n+r}C_{r+1}} = \sum_{n=1}^{\infty} \frac{r+1}{r}\left(\frac{1}{{}^{n+r-1}C_r} - \frac{1}{{}^{n+r}C_r}\right)$ **M1**

$= \frac{r+1}{r}\left(\frac{1}{{}^{r}C_r} - \frac{1}{{}^{r+1}C_r} + \frac{1}{{}^{r+1}C_r} - \frac{1}{{}^{r+2}C_r} + \frac{1}{{}^{r+2}C_r} - \frac{1}{{}^{r+3}C_r} + \cdots\right)$ **M1**

$= \frac{r+1}{r}\frac{1}{{}^{r}C_r}$ because ${}^{n+r}C_r \to \infty$ as $n \to \infty$ **E1**

$= \frac{r+1}{r}$ **A1** **(4)**

$\sum_{n=2}^{\infty} \frac{1}{{}^{n+2}C_{2+1}} = \frac{2+1}{2} - \frac{1}{{}^{1+2}C_{2+1}} = \frac{3}{2} - 1 = \frac{1}{2}$ **M1 M1 (2)**

(ii) ${}^{n+1}C_3 = \frac{(n+1)!}{(n-2)!3!} = \frac{(n+1)n(n-1)}{3!} = \frac{n^3-n}{3!} < \frac{n^3}{3!}$ **M1**

So $\frac{3!}{n^3} < \frac{1}{{}^{n+1}C_3}$ **A1*** **(2)**

$$\frac{20}{{}^{n+1}C_3} - \frac{1}{{}^{n+2}C_5} - \frac{5!}{n^3} = \frac{120}{n(n^2-1)} - \frac{120}{n(n^2-1)(n^2-4)} - \frac{120}{n^3}$$

$= \frac{120}{n^3(n^2-1)(n^2-4)}\left(n^2(n^2-4) - n^2 - (n^2-1)(n^2-4)\right)$ **M1**

$= \frac{-480}{n^3(n^2-1)(n^2-4)} < 0$

as $n \geq 3$ and so denominator is positive. **E1** **(2)**

Hence, $\frac{20}{{}^{n+1}C_3} - \frac{1}{{}^{n+2}C_5} < \frac{5!}{n^3}$

Alternatively,

$$\frac{20}{{}^{n+1}C_3} - \frac{1}{{}^{n+2}C_5} = \frac{5!}{n(n^2-1)} - \frac{5!}{n(n^2-1)(n^2-4)} = \frac{5!}{n(n^2-1)(n^2-4)} \times ((n^2-4)-1)$$

$$= \frac{5!}{n^3} \times \frac{n^4 - 5n^2}{n^4 - 5n^2 + 4} < \frac{5!}{n^3}$$

as $n \geq 3$ and so $n^2 > 5$

$$\sum_{n=3}^{\infty} \frac{3!}{n^3} < \sum_{n=3}^{\infty} \frac{1}{{}^{n+1}C_3} = \sum_{n=2}^{\infty} \frac{1}{{}^{n+2}C_3} = \frac{1}{2}$$

M1

So $\sum_{n=1}^{\infty} \frac{3!}{n^3} < \frac{3!}{1} + \frac{3!}{8} + \frac{1}{2} = \frac{29}{4}$

M1

And therefore $\sum_{n=1}^{\infty} \frac{1}{n^3} < \frac{29}{24} = \frac{116}{96}$

A1* (3)

$$\sum_{n=3}^{\infty} \frac{5!}{n^3} > \sum_{n=3}^{\infty} \left(\frac{20}{{}^{n+1}C_3} - \frac{1}{{}^{n+2}C_5} \right) = 20 \times \frac{1}{2} - \left(\sum_{n=1}^{\infty} \frac{1}{{}^{n+4}C_5} \right) = 10 - \frac{5}{4}$$

M1 **M1**

Therefore

$$\sum_{n=3}^{\infty} \frac{5!}{n^3} > \frac{35}{4}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^3} > \frac{35}{4 \times 5!} + \frac{1}{1} + \frac{1}{8} = \frac{7}{96} + \frac{96}{96} + \frac{12}{96} = \frac{115}{96}$$

M1 **A1* (4)**