STEP III, 2017, Q13 MS

Question 13

To find $\mathrm{V}(x) = \sigma^2 + (x - \mu)^2$, expanding the definition of $\mathrm{V}(x)$ and then expressing $\mathrm{E}(X^2)$ in terms of variance yields the result. The result for $\mathrm{E}(Y)$ that is given in the question follows directly from the first result, and that $\mathrm{V}(x) = \frac{1}{12} + \left(x - \frac{1}{2}\right)^2$ in the uniform case, follows from previous working, applying standard knowledge of the mean and variance of the uniform distribution. In order to find the probability density function of Y, it is simplest to find the cumulative distribution function of Y first which is done by algebraic rearrangement of $\mathrm{P}\left(\frac{1}{12} + \left(X - \frac{1}{2}\right)^2 < y\right)$ and then differentiation to give $\mathrm{f}(y) = \left(y - \frac{1}{12}\right)^{-\frac{1}{2}}$, $\frac{1}{12} \le y \le \frac{1}{3}$ and 0 otherwise. The final verification is conducted by integration of $y\left(y - \frac{1}{12}\right)^{-\frac{1}{2}}$ with suitable limits, which can be done by numerous methods such as expressing the function as $\left(y - \frac{1}{12}\right)\left(y - \frac{1}{12}\right)^{-\frac{1}{2}} + \frac{1}{12}\left(y - \frac{1}{12}\right)^{-\frac{1}{2}}$ or change of variable, of which, letting $u^2 = y - \frac{1}{12}$ is just one example.

$$V(x) = E((X - x)^2) = E(X^2) - 2xE(X) + x^2 = \sigma^2 + \mu^2 - 2x\mu + x^2 = \sigma^2 + (x - \mu)^2$$

M1 M1 M1 A1 (4)

$$E(Y) = E\big(V(X)\big) = E(\sigma^2 + (X - \mu)^2) = \sigma^2 + \sigma^2 = 2\sigma^2$$

M1 A1* (2)

If $X \sim U(0,1)$, then $\mu = \frac{1}{2}$ and $\sigma^2 = \frac{1}{12}$, so $V(x) = \frac{1}{12} + \left(x - \frac{1}{2}\right)^2 = x^2 - x + \frac{1}{3}$

B1 B1 M1 A1 (4)

$$Y = V(X) = X^2 - X + \frac{1}{3} = \frac{1}{12} + \left(X - \frac{1}{2}\right)^2$$

$$Y \in \left[\frac{1}{12}, \frac{1}{3}\right]$$

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$$P(Y<y)=P\left(\frac{1}{12}+\left(X-\frac{1}{2}\right)^2<y\right)=P\left(\frac{1}{2}-\sqrt{y-\frac{1}{12}}<X<\frac{1}{2}+\sqrt{y-\frac{1}{12}}\right)$$

$$=2\sqrt{y-\frac{1}{12}}$$

M1 M1 A1

$$f(y)=\frac{d}{dy}(F(y))=\frac{d}{dy}\left(2\sqrt{y-\frac{1}{12}}\right)=\left(y-\frac{1}{12}\right)^{-\frac{1}{2}},\ \frac{1}{12}\le y\le\frac{1}{3}\text{ and }0\text{ otherwise.}$$

M1 A1 A1 (6)

$$E(Y)=\int_{\frac{1}{12}}^{\frac{1}{3}}y\left(y-\frac{1}{12}\right)^{-\frac{1}{2}}dy=\int_{\frac{1}{12}}^{\frac{1}{3}}\left(y-\frac{1}{12}\right)\left(y-\frac{1}{12}\right)^{-\frac{1}{2}}+\frac{1}{12}\left(y-\frac{1}{12}\right)^{-\frac{1}{2}}dy$$

M1 M1

$$=\left[\frac{2}{3}\left(y-\frac{1}{12}\right)^{\frac{3}{2}}+\frac{1}{6}\left(y-\frac{1}{12}\right)^{\frac{1}{2}}\right]_{\frac{1}{12}}^{\frac{1}{3}}=\frac{1}{12}+\frac{1}{12}=2\times\frac{1}{12}$$

M1 A1 (4)

as required.

Alternatively, for final integral,

let $u^2=y-\frac{1}{12}$,

$$E(Y)=\int_{\frac{1}{12}}^{\frac{1}{3}}y\left(y-\frac{1}{12}\right)^{-\frac{1}{2}}dy=\int_0^{\frac{1}{2}}\frac{u^2+\frac{1}{12}}{u}2u\,du=\left[\frac{2}{3}u^3+\frac{1}{6}u\right]_0^{\frac{1}{2}}=2\times\frac{1}{12}$$

M1 M1 M1 A1 (4)

or further

let $u=y-\frac{1}{12}$,

$$E(Y)=\int_{\frac{1}{12}}^{\frac{1}{3}}y\left(y-\frac{1}{12}\right)^{-\frac{1}{2}}dy=\int_0^{\frac{1}{4}}\frac{u+\frac{1}{12}}{u^{\frac{1}{2}}}du=\left[\frac{2}{3}u^{\frac{3}{2}}+\frac{1}{6}u^{\frac{1}{2}}\right]_0^{\frac{1}{4}}=2\times\frac{1}{12}$$

M1 M1 M1 A1 (4)

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