STEP III, 2017, Q12 MS

Question 12

To obtain the first result of (i), sum in turn over x and y from 1 to n to obtain the total probability (1!) yields $k = \frac{1}{n^2(n+1)}$, and then sum over y. For independence, it would be necessary that $\mathrm{P}(X = x, Y = y) = \mathrm{P}(X = x)\mathrm{P}(Y = y)$ and it is possible to simplify the algebra of this equation having substituted for each probability to see that there are numerous examples for which this does not hold. Proceeding as at the start of the question to obtain $\mathrm{E}(XY) = \frac{(n+1)(2n+1)}{6}$ and summing over x using the printed result of (i) to find $\mathrm{E}(X) = \frac{(7n+5)}{12}$ (and hence $\mathrm{E}(Y)$ also) yields, following simplification, $Cov(X, Y) = \frac{-(n-1)^2}{144}$, establishing the required result.

(i)

$$\sum_{y=1}^{n}\sum_{x=1}^{n} P(X = x, Y = y) = 1$$

$$\sum_{y=1}^{n}\sum_{x=1}^{n} k(x + y) = 1$$

M1

$$k\sum_{y=1}^{n}\left(\frac{1}{2}n(n+1) + ny\right) = 1$$

M1 A1

$$k\left(\frac{1}{2}n^2(n+1) + \frac{1}{2}n^2(n+1)\right) = 1$$

M1

Therefore,

$$k = \frac{1}{n^2(n+1)}$$

A1 (5)

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A1 (5)

$$P(X=x) = \sum_{y=1}^{n} k(x+y) = k\left(nx + \frac{1}{2}n(n+1)\right) = \frac{(2nx + n(n+1))}{2n^2(n+1)} = \frac{n+1+2x}{2n(n+1)}$$

M1 A1 (2)

$$P(Y=y) = \frac{n+1+2y}{2n(n+1)}$$

B1

For X and Y to be independent, $P(X=x, Y=y) = P(X=x) \times P(Y=y)$ **M1**

So

$$\frac{n+1+2x}{2n(n+1)} \times \frac{n+1+2y}{2n(n+1)} = \frac{(x+y)}{n^2(n+1)}$$

M1

$$(n+1+2x)(n+1+2y) = 4(n+1)(x+y)$$

$$(n+1)^2 - 2(n+1)(x+y) + 4xy = 0$$

$$\left((n+1)-(x+y)\right)^2 - (x-y)^2 = 0$$

M1

which does not happen for e.g. $x = n$, $y = 1$. (Many equally valid examples possible.)

X and Y are not independent. **E1 (5)**

(ii)

$$E(XY) = \sum_{y=1}^{n}\sum_{x=1}^{n} kxy(x+y) = k\sum_{y=1}^{n}\left(y\frac{n(n+1)(2n+1)}{6} + y^2\frac{n(n+1)}{2}\right)$$

M1

$$= k\frac{n^2(n+1)^2(2n+1)}{6} = \frac{(n+1)(2n+1)}{6}$$

M1 A1 (3)

$$E(X) = E(Y) = \sum_{x=1}^{n} x\frac{n+1+2x}{2n(n+1)} = \frac{\frac{1}{2}n(n+1)^2 + \frac{2}{6}n(n+1)(2n+1)}{2n(n+1)}$$

M1 A1

$$= \frac{(n+1)}{4} + \frac{(2n+1)}{6} = \frac{(7n+5)}{12}$$

A1 (3)

Thus

$$Cov(X,Y) = \frac{(n+1)(2n+1)}{6} - \left(\frac{(7n+5)}{12}\right)^2 = \frac{-n^2+2n-1}{144} = \frac{-(n-1)^2}{144} < 0$$

M1 E1 (2)