STEP III, 2017, Q1

1 **(i)** Prove that, for any positive integers n and r,

$$\frac{1}{{}^{n+r}\mathrm{C}_{r+1}} = \frac{r+1}{r}\left(\frac{1}{{}^{n+r-1}\mathrm{C}_r} - \frac{1}{{}^{n+r}\mathrm{C}_r}\right).$$

Hence determine

$$\sum_{n=1}^{\infty} \frac{1}{{}^{n+r}\mathrm{C}_{r+1}},$$

and deduce that $\displaystyle\sum_{n=2}^{\infty} \frac{1}{{}^{n+2}\mathrm{C}_3} = \frac{1}{2}$.

(ii) Show that, for $n \geqslant 3$,

$$\frac{3!}{n^3} < \frac{1}{{}^{n+1}\mathrm{C}_3} \quad \text{and} \quad \frac{20}{{}^{n+1}\mathrm{C}_3} - \frac{1}{{}^{n+2}\mathrm{C}_5} < \frac{5!}{n^3}.$$

By summing these inequalities for $n \geqslant 3$, show that

$$\frac{115}{96} < \sum_{n=1}^{\infty} \frac{1}{n^3} < \frac{116}{96}.$$

Note: ${}^{n}\mathrm{C}_r$ is another notation for $\dbinom{n}{r}$.