

## STEP III, 2016 , Q8 MS

8. The first result in (i) is obtained by the substitution  $x = -u$  (followed by a second  $u = x - 1$ ). Substituting for  $f(-x)$  in the initial statement using the result obtained readily leads to  $f(x) = x$  which is simply verified. Alternatively, subtracting the result from the initial equation leads to  $f(x) = f(-x)$  which substituting gives the required result again. In part (ii), substituting  $K(x)$  for  $x$  in the equation for  $g(x)$  gives an equation for  $g\left(\frac{x+1}{x-1}\right)$  which can be substituted in the equation to be solved to give the desired result. Similarly, in part (iii), substituting  $\frac{1}{1-x}$  for  $x$  gives an equation for  $h\left(\frac{1}{1-x}\right)$  and  $h\left(\frac{x-1}{x}\right)$ , and then repeating this substitution in the equation just obtained gives an equation for  $h\left(\frac{x-1}{x}\right)$  and  $h(x)$ . Adding the given and last equations and subtracting that first found leads to  $h(x) = \frac{1}{2} - x$ .



# NextStepMaths.com

To view mark schemes, fully worked solutions and examiner's comments, and for more details about tutoring and other services offered, go to [NextStepMaths.com](https://www.NextStepMaths.com)

8. (i)  $f(x) + (1 - x)f(-x) = x^2$

Let  $x = -u$ , then  $f(-u) + (1 - -u)f(- - u) = (-u)^2$

i.e.  $f(-u) + (1 + u)f(u) = u^2$

Let  $u = x$ , then  $f(-x) + (1 + x)f(x) = x^2$  as required. **E1**

Substituting for  $f(-x)$  from the equation just obtained in the original, **M1**

$$f(x) + (1 - x)(x^2 - (1 + x)f(x)) = x^2$$

Thus  $x^2f(x) = x^3$ , and hence  $f(x) = x$  **M1 A1**

Verification:-  $x + (1 - x) \times -x = x - x + x^2 = x^2$  as required. **B1 (5)**

(ii)

$$K(K(x)) = K\left(\frac{x+1}{x-1}\right) = \frac{\left(\frac{x+1}{x-1}\right) + 1}{\left(\frac{x+1}{x-1}\right) - 1} = \frac{x+1+x-1}{x+1-x+1} = \frac{2x}{2} = x$$

**M1 M1 A1\* (3)**

as required.

$$g(x) + xg\left(\frac{x+1}{x-1}\right) = x$$

So

$$g\left(\frac{x+1}{x-1}\right) + \left(\frac{x+1}{x-1}\right)g\left(\frac{\left(\frac{x+1}{x-1}\right) + 1}{\left(\frac{x+1}{x-1}\right) - 1}\right) = \left(\frac{x+1}{x-1}\right)$$

**M1**

That is

$$g\left(\frac{x+1}{x-1}\right) + \left(\frac{x+1}{x-1}\right)g(x) = \left(\frac{x+1}{x-1}\right)$$

**A1**

So substituting for  $g\left(\frac{x+1}{x-1}\right)$  from the equation just obtained in the initial equation **M1**

$$g(x) + x\left(\left(\frac{x+1}{x-1}\right) - \left(\frac{x+1}{x-1}\right)g(x)\right) = x$$

$$[(x-1) - x(x+1)]g(x) + x(x+1) = x(x-1)$$

$$(-x^2 - 1)g(x) = -2x$$



# NextStepMaths.com

To view mark schemes, fully worked solutions and examiner's comments, and for more details about tutoring and other services offered, go to [NextStepMaths.com](http://NextStepMaths.com)

$$g(x) = \frac{2x}{(x^2 + 1)}$$

**M1 A1\* (5)**

Not required - verification:-

$$\begin{aligned} \frac{2x}{(x^2 + 1)} + x \frac{2 \left( \frac{x+1}{x-1} \right)}{\left( \left( \frac{x+1}{x-1} \right)^2 + 1 \right)} &= \frac{2x}{(x^2 + 1)} + x \left( \frac{2(x+1)(x-1)}{(x+1)^2 + (x-1)^2} \right) = \frac{2x}{(x^2 + 1)} + x \frac{2(x^2 - 1)}{2(x^2 + 1)} \\ &= \frac{2x + x(x^2 - 1)}{(x^2 + 1)} = \frac{x(2 + x^2 - 1)}{(x^2 + 1)} = x \end{aligned}$$

as expected.

(iii)

$$h(x) + h\left(\frac{1}{1-x}\right) = 1 - x - \frac{1}{1-x}$$

(Equation A)

$$h\left(\frac{1}{1-x}\right) + h\left(\frac{1}{1 - \left(\frac{1}{1-x}\right)}\right) = 1 - \left(\frac{1}{1-x}\right) - \frac{1}{1 - \left(\frac{1}{1-x}\right)}$$

**M1 A1**

Thus

$$h\left(\frac{1}{1-x}\right) + h\left(\frac{x-1}{x}\right) = 1 - \left(\frac{1}{1-x}\right) + \left(\frac{1-x}{x}\right)$$

(Equation B)

Then

$$h\left(\frac{x-1}{x}\right) + h\left(\frac{1}{1 - \left(\frac{x-1}{x}\right)}\right) = 1 - \left(\frac{x-1}{x}\right) - \frac{1}{1 - \left(\frac{x-1}{x}\right)}$$

**M1 A1**

That is

$$h\left(\frac{x-1}{x}\right) + h(x) = 1 - \left(\frac{x-1}{x}\right) - x$$

(Equation C)



# NextStepMaths.com

To view mark schemes, fully worked solutions and examiner's comments, and for more details about tutoring and other services offered, go to [NextStepMaths.com](https://www.NextStepMaths.com)

A+C-B gives

$$2h(x) = 1 - x - \frac{1}{1-x} + 1 - \left(\frac{x-1}{x}\right) - x - \left(1 - \left(\frac{1}{1-x}\right) + \left(\frac{1-x}{x}\right)\right)$$

**M1**

**A1**

$$2h(x) = 1 - 2x$$

So

$$h(x) = \frac{1}{2} - x$$

**A1 (7)**

Not required - verification:-

$$\frac{1}{2} - x + \frac{1}{2} - \frac{1}{1-x} = 1 - x - \frac{1}{1-x}$$

as expected.



# NextStepMaths.com

To view mark schemes, fully worked solutions and examiner's comments, and for more details about tutoring and other services offered, go to [NextStepMaths.com](https://www.NextStepMaths.com)