STEP III, 2016 , Q7 MS

7. Considering $(\omega^r)^n$ establishes by the factor theorem that each factor on the LHS is a factor of the RHS, and comparing coefficients of z^n between the two sides establishes that no numerical factor is required. For part (i), representing X_r by ω^r , then there are two cases to consider, P will be represented either by $re^{\frac{\pi i}{n}}$, or $re^{\left(\frac{\pi}{n}+\pi\right)i}$. The product of moduli is the moduli of the product of factors, and the product of the factors can be simplified using the stem and choosing z in turn as the representations of P to give the required result in both cases. Proceeding similarly for *n* odd, the first case yields $|OP|^n + 1$, and the second, $|OP|^n - 1$, if $|OP| \geq 1$, and $1 - |OP|^n$ if $|OP| < 1$. Using the same representations for the X_r in part (ii), and the same technique with the moduli, the stem can be divided by $(z-1)$ to give $(z-\omega)(z-\omega^2)\dots(z-\omega^{n-1}) = z^{n-1} + z^{n-2} + \cdots + 1$ which then gives the desired result when $z = 1$.

7. If

$$\omega = e^{\frac{2\pi i}{n}}$$

then if $0 \le r \le n-1$,

$$(\omega^r)^n = e^{\frac{2\pi i r n}{n}} = \left(e^{2\pi i}\right)^r = 1^r = 1$$

M1

So $1, \omega, \omega^2, \dots, \omega^{n-1}$ are the n roots of $z^n = 1$, that is of $z^n - 1 = 0$. **A1**

Thus $(z - \omega^r)$ is a factor of $z^n - 1$ **B1**

Hence $z^n - 1 = k(z-1)(z-\omega)(z-\omega^2)\dots(z-\omega^{n-1})$ and comparing coefficients of z^n , $k = 1$

M1

So as required $(z-1)(z-\omega)(z-\omega^2)\dots(z-\omega^{n-1}) = z^n - 1$ **A1* (5)**

(i) Without loss of generality, let X_r be represented by ω^r **M1**

Then P will be represented either by $re^{\frac{\pi i}{n}} = z$, or $re^{\left(\frac{\pi}{n}+\pi\right)i} = z'$ with $|OP| = r$ **M1**

$$|PX_0| \times |PX_1| \times \dots \times |PX_{n-1}| = \left|1 - re^{\frac{\pi i}{n}}\right| \left|\omega - re^{\frac{\pi i}{n}}\right| \dots \left|\omega^{n-1} - re^{\frac{\pi i}{n}}\right|$$

M1

$$= |(z-1)(z-\omega)(z-\omega^2)\dots(z-\omega^{n-1})| = |z^n - 1| = \left|r^n e^{\pi i} - 1\right| = |-r^n - 1| = r^n + 1$$

A1 **A1** **A1***

or $|z'^n - 1| = \left|r^n e^{(n+1)\pi i} - 1\right| = \left|r^n e^{\pi i} e^{n\pi i} - 1\right| = |-r^n - 1| = r^n + 1$ as $e^{n\pi i} = 1$ because n is even. **E1 (7)**

So $|PX_0| \times |PX_1| \times \dots \times |PX_{n-1}| = |OP|^n + 1$ as required.

For n odd,

$|PX_0| \times |PX_1| \times \dots \times |PX_{n-1}| = |z^n - 1| = \left|r^n e^{\pi i} - 1\right| = |-r^n - 1| = r^n + 1 = |OP|^n + 1$

M1 A1

or

$$|PX_0| \times |PX_1| \times \dots \times |PX_{n-1}| = \left|z'^n - 1\right| = \left|r^n e^{(n+1)\pi i} - 1\right| = |r^n - 1| = |OP|^n - 1$$

B1

if $|OP| \ge 1$, and $= 1 - |OP|^n$ if $|OP| < 1$ **A1 (4)**

(ii)

$$|X_0X_1| \times |X_0X_2| \times \ldots \times |X_0X_{n-1}| = |(1-\omega)(1-\omega^2)\ldots(1-\omega^{n-1})|$$

M1

But

$$(z-1)(z-\omega)(z-\omega^2)\ldots(z-\omega^{n-1}) = z^n - 1$$

and so

$$(z-\omega)(z-\omega^2)\ldots(z-\omega^{n-1}) = \frac{z^n-1}{z-1} = z^{n-1} + z^{n-2} + \cdots + 1$$

M1

$$(z-\omega)(z-\omega^2)\ldots(z-\omega^{n-1}) = z^{n-1} + z^{n-2} + \cdots + 1$$

A1

is true for all z so for $z = 1$, $(1-\omega)(1-\omega^2)\ldots(1-\omega^{n-1}) = 1 + 1 + \cdots + 1 = n$ **A1* (4)**